NOTES ON DRINFELD'S PROOF OF SERRE-TATE

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1. Drinfeld's proof of Serre-Tate

1.1. **Preliminaries.** Let R be a ring which is a $\mathbf{Z}/N$-algebra for some $N \geq 1$ and I an ideal which is nilpotent i.e. $I^{\nu+1} = 0$ (note that this implies that R is I-complete). We denote by R_0 the ring R/I. Note that $\pi\colon \operatorname{Spec} R_0 \to \operatorname{Spec} R$ is a closed immersion with the same underlying topological space.

Definition 1.1. Let $G\colon \operatorname{Alg}_R \to \operatorname{Grp}$ we denote have the two functors G_I and $\widehat{G}$ also on Alg_R defined as

$$G_I(A) = \ker(G(A) \to G(A/I))$$

and

$$\widehat{G}(A) = \ker(G(A) \to G(A^{\mathrm{red}})).$$

Remark 1.2. Observe that $I \subset \operatorname{Nil}(A)$ and so there is a map $G(A/I) \to G(A^{\mathrm{red}})$ factoring the canonical map $G(A) \to G(A^{\mathrm{red}})$. In particular $G_I \subset \widehat{G}$ as subfunctors.

Remark 1.3. Note that $\widehat{G}(A/I) = \ker(G(A/I) \to G((A/I)^{\mathrm{red}}))$ but the target is $G(A^{\mathrm{red}})$ so the containment in Remark 1.2 factors as an isomorphism $G_I \simeq (\widehat{G})_I$.

Remark 1.4. We are mostly interested in the case that G is either an abelian scheme or a formal Lie group over R.

Definition 1.5. For our purposes, a formal Lie group over R is a 'framed' formally smooth scheme of the form $\operatorname{Spf} R[[x_1, \ldots, x_n]]$ with a Hopf algebra structure on its coordinate ring.

We begin by observing that if G is a formal Lie group then G_I is N^ν-torsion.

Lemma 1.6. *If G is a commutative formal Lie group then $N^\nu G_I = 0$.*

Proof. Recall first that $G = \operatorname{Spf} R[[x_1, \ldots, x_n]]$ and so its A points for any R-algebra A are given by $\underline{a} := (a_1, \ldots, a_n) \in A$. If $\underline{a} \in G_I$ then it follows that $a_i \in IA$ for all i.

Now observe that

$$[N](\underline{a})_i = Na_i + \text{ higher order terms}$$

where the higher order terms depend on the Hopf algebra structure on $\mathcal{O}_G$.

Now since A is also N-torsion we see that $[N]\underline{a} \in I^2$ and so $[N]G_I \subset G_{I^2}$. Doing this ν-times we observe that $[N^\nu]G_I \subset G_{I^{2\nu}}$ but $I^{2\nu} \subset I^\nu$ and since $G_{I^\nu} = 0$ we get the result.

□

Corollary 1.7. *Let G be an abelian sheaf on $(\operatorname{Alg}_R)_{fppf}$, such that fppf locally $\widehat{G}$ is representable by a Lie group. Then $N^\nu G_I = 0$.*

Proof. By Remarks 1.2 and 1.3 it follows that $G_I \subset \widehat{G}$ and so $G_I = (\widehat{G})_I$ while the latter is locally 0 by Lemma 1.6 thence 0. □

Notation 1.8. If G is any functor over Alg_R, we denote by G_0 its pullback to Alg_{R_0}.

For our purposes we recall the definition of an N-divisible abelian sheaf on $(\operatorname{Alg}_R)_{\mathrm{fppf}}$.

Definition 1.9. An abelian sheaf $G\colon (\operatorname{Alg}_R)_{\mathrm{fppf}} \to \operatorname{Ab}$ is N-divisible if $[N]\colon G \to G$ is fppf surjective.

Remark 1.10. Note that this is weaker than the definition of a p-divisible (Barsotti-Tate) group where $N = p$. In particular we don't require any representability nor flatness (as a $\mathbf{Z}/p$-module) conditions on $G[p]$.

In the sequel, we are interested in the question of deformations of morphisms. Namely, let G and H be two N-divisible groups over R and G_0 and H_0 as in 1.8 then there is an obvious reduction map

$$\mathrm{Hom}_{\mathrm{Ab}(R_{\mathrm{fppf}})}(G, H) \to \mathrm{Hom}_{\mathrm{Ab}((R_0)_{\mathrm{fppf}})}(G_0, H_0). \tag{1.1}$$

where $\mathrm{Ab}(R_{\mathrm{fppf}})$ is the large category of abelian sheaves on the fppf topology on Alg_R.

One can ask if the map of 1.1 is a lossless procedure. As it turns out, the answer is affirmative with even milder hypothesis on G and H.

For the remainder of this subsection, we assume the following hypothesis holds.

Hypothesis 1.11. Let G and H be two abelian fppf sheaves on Alg_R.

(1) G is N-divisible,
(2) $\widehat{H}$ is locally representable by a formal Lie group.
(3) H is formally smooth.

Remark 1.12. If G is either an abelian scheme or the connected Lie group associated to a Barsotti-Tate group over R, then it satisfies the hypothesis in 1.11.

Lemma 1.13. *The groups* $\mathrm{Hom}_{\mathrm{Ab}(R_{\mathrm{fppf}})}(G, H)$ *and* $\mathrm{Hom}_{\mathrm{Ab}((R_0)_{\mathrm{fppf}})}(G_0, H_0)$ *have no* N*-torsion.*

Proof. Since G is N-divisible, $[N]\colon G \to G$ is fppf surjective. Now any $f : G \to H$ commutes with $[N]$. It follows that if $[N] \circ f = 0$ then $f = 0$. □

Lemma 1.14. *The map of 1.1 is injective.*

Proof. Note the sequence of presheaves (hence sheaves) given by

$$0 \to H_I \to H \to H_0$$

is exact. Then apply $\mathrm{Hom}_{\mathrm{Ab}(R_{\mathrm{fppf}})}(G, -)$ and note that any morphis $G \to H_0$ factors through G_0. This shows that the kernel of 1.1 is the group $\mathrm{Hom}_{\mathrm{Ab}(R_{\mathrm{fppf}})}(G, H_I)$. Since the source is N-divisible and the target is N^ν-torsion by 1.7, it follows that the kernel is 0. □

Lemma 1.15. *For any morphism* $f_0\colon G_0 \to H_0$ *there exists a unique morphism* '$N^\nu f$'$\colon G \to H$ *which lifts* $N^\nu f_0\colon G_0 \to H_0$.

Proof. The map of 1.1 is injective by 1.14. It follows that it is enough to write down a lift which is then unique. By formal smoothness of H (see 1.11) it follows that the morphism $H(A) \to H(A/I)$ is surjective. The lifts are a torsor over $H_I(A)$, a group which is killed by N^ν by 1.6. Thus we may lift $N^\nu f_0$ by $N^\nu \times g$ where g is any lift of f_0. By uniqueness, this is the requisite lift. □

Lemma 1.16. *Given a morphism* $f_0\colon G_0 \to H_0$ *a lift* $f\colon G \to H$ *exists if and only if* '$N^\nu f$' *annihilates* $G[N^\nu] = \ker(G \xrightarrow{N^\nu} G)$.

Proof. Suppose such a lift exists, then $N^\nu f$ lifts $N^\nu f_0$ and by 1.15 agrees with '$N^\nu f$'. This clearly annihilates $G[N^\nu]$.

For the converse suppose '$N^\nu f$' annihilates $G[N^\nu]$. Then '$N^\nu f$' factors as $G \to G/G[N^\nu] \to H$. The middle term is again G (by N-divisibility) and so '$N^\nu f$' is of the form $N^\nu F$ for some $F \in \mathrm{Hom}_{\mathrm{Ab}(R_{\mathrm{fppf}})}(G, H)$. Now $N^\nu F_0$ agrees with $N^\nu f_0$ but so $F_0 = f_0$ by 1.13. □

1.2. **Serre-Tate.** In this section we specialize to the case where $N = p^n$ for some $n \geq 1$.

Let AbSch_R and AbSch_{R_0} denote the category of abelian schemes over R and R_0 respectively. We will consider the deformation problem along the canonical base change $\mathrm{AbSch}_R \to \mathrm{AbSch}_{R_0}$.

Definition 1.17. Let $\mathrm{Def}(R, R_0)$ be the category consisting of triples (A_0, G, ε) where

(1) A_0 is an abelian scheme over R_0,
(2) G is a Barsotti-Tate group over R,
(3) and $\varepsilon\colon G_0 \xrightarrow{\sim} A_0[p^\infty]$ is an isomorphism of Barsotti-Tate groups.

Construction 1.18. The functor $\mathrm{AbSch}_R \to \mathrm{AbSch}_{R_0}$ refines to a natural functor $\mathrm{AbSch}_R \to \mathrm{Def}(R, R_0)$ sending

$$A \mapsto (A_0, A[p^\infty], \mathrm{can})$$

where can is the natural isomorphism $A[p^\infty]_0 \simeq A_0[p^\infty]$ given by base change.

In the sequel we study the functor of Construction 1.18 and our goal is to prove it is an equivalence (see Theorem 1.22).

We break down the proof of into two parts. We will use the observation in Remark 1.12 and it's subsequent consequences.

Proposition 1.19. *The functor in 1.18 is fully faithful.*

Proof. We are given the following pieces of data

(1) Abelian varieties A and B over R.
(2) A morphism $f[p^\infty]\colon A[p^\infty] \to B[p^\infty]$ of p-divisible groups over R
(3) A morphism $f_0\colon A_0 \to B_0$ with the property that $f_0[p^\infty] = f[p^\infty]_0$.

We need to show that there is a unique morphism $f\colon A \to B$ over R which induces both $f[p^\infty]$ and f_0 with the requisite property.

By Lemma 1.14, we just need to show existence.

First note that by Lemma 1.15 there exists a lift '$N^\nu f'\colon A \to B$ lifting $N^\nu f_0\colon A_0 \to B_0$. Recall here that $N = p^n$ for some $n \geq 1$. Therefore, by Lemma 1.16 it remains to show that '$N^\nu f'\colon A \to B$ annihilates $A[N^\nu]$. We can take '$N^\nu f'[p^\infty]\colon A[p^\infty] \to B[p^\infty]$ to be the associated morphism on p-divisible groups. Since we are already given a morphism $f[p^\infty]\colon A[p^\infty] \to B[p^\infty]$ this implies again by Lemmas 1.14 and 1.15 that

$$`N^\nu f'[p^\infty] = N^\nu f[p^\infty].$$

But this means that $'N^\nu f'$ annihilates $A[N^\nu]$ and so again by applications of Lemmas 1.16 and 1.14 we win. $\square$

It remains to show essential surjectivity.

Remark 1.20. We first recall the following fact from the obstruciton theory of Abelian schemes over a perfect field of positive characteristic. Let $S \to S_0$ be a nilpotent thickening with ideal I of algebras over $\mathbf{Z}/p^n$. Then the obstructions to lifting an abelian scheme $f\colon A_0 \to S_0$ is given by a class $\omega(S, f_0) \in \mathrm{Ext}^2(L_{A_0/S_0}, f_0^* I) = H^2(A_0, T_{A_0/S} \otimes f_0^* I)$ vanish (even though the groups don't unless the relative dimension is 1).

Proposition 1.21. *The functor of 1.22 is essentially surjective.*

Proof. We need to show that given a triple (A_0, G, ε) as in 1.17, we need to find an Abelian variety A/R that induces the triple.

By Remark 1.20 we can find an abelian variety B/R so that $\alpha_0\colon B_0 \simeq A_0$ as abelian schemes over R_0.

We have the morphism

$$\alpha_0[p^\infty]\colon B_0[p^\infty] \to A_0[p^\infty].$$

Then by Lemma 1.15 we have a morphism '$N^\nu\alpha'\colon B[p^\infty] \to G$ lifting $N^\nu\alpha_0$. We cannot deduce from this that there exists an $\alpha\colon B[p^\infty] \to G$, but we can deduce that the morphism '$N^\nu\alpha'$ is an isogeny. Ineed, an inverse upto isogeny is given by $'N^\nu(\alpha^{-1})'\colon G \to B[p^\infty]$ and the composite '$N^\nu\alpha' \circ' N^\nu(\alpha^{-1})'$ is multiplication by $N^{2\nu}$ which is fppf surjective. In particular we have a short exact sequence

$$0 \to K \to B[p^\infty] \xrightarrow{`N^\nu \times \alpha'} G \to 0$$

where $K \subset B[N^{2\nu}]$ and hence finite.

If K was flat then we would be done since the abelian sheaf B/K would be representable by an abelian scheme which clearly solves the problem.

So it is enough to show that '$N^\nu \times \alpha'$ is flat. Both the source and the target are flat and any fibers of the morphism factor through $\operatorname{Spec} R_0 \to \operatorname{Spec} R$. Thus it is enough to check that the morphism is flat mod I, in which case it is $N^\nu\alpha_0$ which is multiplication by N^ν (which is flat) composed with α_0 which is an isomorphism. $\square$

We can now prove Theorem 1.22.

Theorem 1.22. *(Serre-Tate) The functor of 1.18 is an equivalence of categories.*

Proof. Follows from Propositions 1.19 and 1.21. □

Remark 1.23. Note that the assertion in Theorem 1.22 is very powerful. The deformation problem in Definition 1.17 only knows about the Barsotti-Tate group of an abelian scheme A_0/R_0 and is asserting that solving that deformation problem is *equivalent* to lifting A_0/R_0 to an abelian scheme over A/R.