On the (co-) homology of commutative rings

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This paper is devoted to a summary of the main results of a (co-) homology theory of commutative rings due independently to André [1] and the author [19]. The theory has reached a degree of completeness that there is little doubt that it is the correct cohomology theory for commutative rings. It is perhaps worthwhile to relate this theory to definitions made by other authors.

For any morphism $A \to B$ of commutative rings and B-module M, let $\mathrm{Der}_A(B, M)$ be the B-module of A-algebra derivations of B with values in M, and let $\mathrm{Exalcomm}_A(B, M)$ be the B-module of infinitesimal A-algebra extensions of B by M ([11, Chapter IV]). It is known that the functors Der and Exalcomm possess the following two properties:

0.1. (*Transitivity*). Given morphisms $A \to B \to C$ of commutative rings and a C-module M there is a six-term exact sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M)$$
$$\to \mathrm{Exalcomm}_B(C, M) \to \mathrm{Exalcomm}_A(C, M) \to \mathrm{Exalcomm}_A(B, M).$$

0.2. (*Flat base change*). Given morphisms $A \to B$, $A \to A'$ of commutative rings and an $A' \otimes_A B$ module M, there are isomorphisms

$$\mathrm{Der}_{A'}(A' \otimes_A B, M) \simeq \mathrm{Der}_A(B, M)$$
$$\mathrm{Exalcomm}_{A'}(A' \otimes_A B, M) \simeq \mathrm{Exalcomm}_A(B, M) \quad \text{if } \mathrm{Tor}_1^A(A', B) = 0.$$

The cohomology theory to be presented here associates to each morphism $A \to B$ of commutative rings and B-module M cohomology B-modules $D^q(B/A, M)$ for each integer $q \geq 0$, coinciding with $\mathrm{Der}_A(B, M)$ and $\mathrm{Exalcom}_A(B, M)$ for $q = 0$ and 1 respectively, and extending in the obvious way the properties 0.1–0.2.

In [13] Harrison gave a definition of commutative ring cohomology using a

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subcomplex of the Hochschild complex for computing associative algebra cohomology. Harrison restricted himself to the case where the ground ring A is a field; his definition does not extend to a general morphism $A \to B$ since one needs at least that B be projective as an A-module in order that Harrison's H^2 coincide with $\text{Exalcomm}_A(B, M)$. However, it can be shown (§9 below) that Harrison's complex may be used to compute the correct cohomology $D^*(B/A, M)$ when B is projective as an A-module and when B is of characteristic 0, that is, an algebra over the rational numbers.

In [**16**], Lichtenbaum and Schlessinger define a satisfactory cohomology theory in dimensions $q \leq 2$ in the sense that their cohomology extends the exact sequence of 0.1 to nine terms. Their method uses a free differential graded anticommutative A-algebra resolution of B. It may be shown that their cohomology group $T^q(B/A, M)$ coincides with our $D^q(B/A, M)$ for $q \leq 2$ and that free differential graded anticommutative A-algebra resolutions of B may be used to compute $D^*(B/A, M)$ when B is of characteristic zero (§9).

In [**21**] it is shown how the transitivity and flat base change properties of the Lichtenbaum-Schlessinger $T^q(B/A, M)$ for $q \leq 2$ (hence also of our $D^q(B/A, M)$) yield a satisfactory deformation theory for commutative algebras. Moreover the Lichtenbaum-Schlessinger cohomology is known to be the same as the theory of Gerstenhaber [**8**] specialized to commutative rings.

The cohomology groups $D^*(B/A, M)$ are defined as suitable nonabelian derived functors of the functor $X \mapsto \text{Der}_A(X, M)$ on the category of A-algebras over B, where the derived functors are defined by applying the functor dimensionwise to a free (semi-) simplicial A-algebra resolution of B and taking the cohomology of the associated cosimplicial B-module. A particular example of such a resolution is furnished by the standard construction associated to the (sets)—(A-algebras) cotriple. Thus the cohomology $D^*(B/A, M)$ is a kind of cotriple cohomology [**5**], however, from the cotriple point of view one loses sight of the possibility of choosing a resolution for computing the derived functors with special properties, a technique vital for proving the transitivity and flat base change properties of the commutative ring cohomology.

1. **Derived functors.** In a category closed under finite limits and having enough projective objects, it is possible to define left derived functors of any functor from the category to an abelian category generalizing those of Dold-Puppe [7]. We give the construction of these derived functors in this section.

An object P of a category $\mathscr{C}$ is said to be *projective* if for any effective epimorphism $p: X \to Y$, $\text{Hom}(P, p): \text{Hom}(P, X) \to \text{Hom}(P, Y)$ is surjective. $\mathscr{C}$ is said to have *enough projective objects* if for every object X there is an effective epimorphism $P \to X$ with P projective. When $\mathscr{C}$ is closed under finite projective limits and has enough projective objects, then the effective epimorphisms are the maps p for which $\text{Hom}(P, p)$ is surjective for all projective objects P [**18,** II, §4, Proposition 2]; in particular the effective epimorphisms are universally effective.

Let $s\mathscr{C}$ be the category of (semi-) simplicial objects over $\mathscr{C}$. We always identify an object X of $\mathscr{C}$ with the constant simplicial object whose simplicial operators are all equal to the identity morphism of X. For X, $Y \in \operatorname{Ob} s\mathscr{C}$, let $\operatorname{Hom}(X, Y)$ be the "function complex" simplicial set of morphisms from X to Y; a homotopy from a morphism $f: X \to Y$ to g is thus an element h of $\operatorname{Hom}(X, Y)_1$ with $d_1 h = f$ and $d_0 h = g$. Assuming $\mathscr{C}$ is closed under finite limits, it is possible to define **[18,** II, §1, Proposition 2] for each object X of $s\mathscr{C}$ and simplicial set K having only finitely many nondegenerate simplices generalized "cylinder" and "function-space" objects X_K and X^K satisfying the formulas

$$\text{(1.1)} \qquad \operatorname{Hom}(X_K, Y) = \operatorname{Hom}_{s(sets)}(K, \operatorname{Hom}(X, Y)) = \operatorname{Hom}(X, Y^K).$$

In particular if I is the standard 1-simplex, there are canonical maps $d_0^X, d_1^X : X^I \to X$ such that to give a simplicial homotopy joining $f: Y \to X$ to g is the same as giving a map $h: Y \to X^I$ such that $d_1 h = f$ and $d_0 h = g$. Also there is a canonical map s_0: $X \to X^I$ representing the constant homotopy from id_K to itself.

We say that a morphism $i: A \to B$ in $\mathscr{C}$ has the *left lifting property* (LLP) with respect to a morphism $p: X \to Y$, or that p has the *right lifting property* (RLP) with respect to i, if the dotted arrow exists in any commutative square of solid arrows

$$\text{(1.2)} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

A morphism of simplicial sets will be called an *acyclic fibration* ("acyclic" is much preferable to the term "trivial" used in **[18]**) if it has the RLP with respect to any monomorphism of simplicial sets, or equivalently if it is a Kan fibration which is surjective and whose fibers are contractible. A morphism of simplicial objects over a category $\mathscr{C}$ will be called an *acyclic fibration* if for any projective object P, the induced morphism $\operatorname{Hom}(P, X) \to \operatorname{Hom}(P, Y)$ is an acyclic fibration of simplicial sets. Finally we call a morphism in $s\mathscr{C}$ a *cofibration* if it has the LLP with respect to all acyclic fibrations.

PROPOSITION 1.3. *Assume $\mathscr{C}$ is closed under finite projective limits and let $i: A \to B$ be a cofibration and $p: X \to Y$ an acyclic fibration in $s\mathscr{C}$. Then given a commutative square* 1.2 *any two choices for the dotted arrow are homotopic under A and over B. More generally given homotopies $\bar{\alpha}: A \to X^I$, $\bar{\beta}: B \to Y^I$ compatible with i and p (i.e. $p^I \bar{\alpha} = \bar{\beta} i$), and given dotted arrows θ, $\bar{\theta}: B \to X$ with $\theta i = d_1 \bar{\alpha}$, $\theta' i = d_0 \bar{\alpha}$, $p\theta = d_1 \bar{\beta}$, $p\theta' = d_0 \bar{\beta}$, there is a homotopy $\bar{\theta}: B \to X^I$ joining θ to θ' compatible with $\bar{\alpha}$ and $\bar{\beta}$ (i.e. $\bar{\theta} i = \bar{\alpha}$ and $p^I \bar{\theta} = \bar{\beta}$).*

$\bar{\theta}$ is obtained as the dotted arrow in the square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X^I \arrow[d, "(d_1^X, d_0^X, p^I)"] \\ B \arrow[r, "(\theta, \theta', \beta)"'] & X^{\dot{I}} \times_{(Y^{\dot{I}})} (Y^I) \end{tikzcd}$$

where $X^I = X \times X$. To see that (d_1^X, d_0^X, p^I) is an acyclic fibration one applies the functor Hom $(P, ?)$, P a projective object of $\mathscr{C}$; using 1.1 one is reduced to the case of simplicial sets which may then be handled by means of the formula $L_K = L \times K$.

PROPOSITION 1.4. *Let $\mathscr{C}$ be a category closed under finite limits and having sufficiently many projective objects. Then any morphism f of $s\mathscr{C}$ may be factored $f = pi$ where i is cofibrant and p is an acyclic fibration.*

This factorization is constructed by a simplicial analogue of the familiar process in algebraic topology of killing homotopy groups by attaching cells. For details see [**18**, II, §4, Proposition 3]. By standard arguments using 1.3, the factorization is unique up to homotopy equivalence, and up to homotopy depends functorially on the morphism f.

Suppose now that $\mathscr{C}$ satisfies the hypotheses of 1.4. If X is an object of $\mathscr{C}$, we may define a simplicial *resolution* of X to be an acyclic fibration $Q \to X$. A simplicial object P will be called *cofibrant* if the map $\phi \to P$ is a cofibration where ϕ is the initial object of $\mathscr{C}$. (The term "projective" instead of cofibrant is perhaps more in keeping with tradition except that a cofibrant P need not be projective in the category $s\mathscr{C}$.) By 1.3 and 1.4 cofibrant resolutions of X exist, are unique up to homotopy equivalence, and up to homotopy depend functorially on X. Therefore if $F:\mathscr{C} \to \mathscr{A}$ is a functor, where $\mathscr{A}$ is an abelian category, we may define left derived functors by $(L_nF)(X) = H_nF(P)$, where as usual the homology of a simplicial object in an abelian category is the homology of the associated chain complex with $d = \Sigma(-1)^i d_i$, or of the normalized subcomplex.

If $\mathscr{C}$ is an abelian category with enough projectives, then by [7] the normalization $N: s\mathscr{C} \to \mathrm{Ch}\,(\mathscr{C})$ is an equivalence of $s\mathscr{C}$ with the category of chain complexes of $\mathscr{C}$. One may show (see [**18**, II, §3, Proposition 3]) that a map $f:X \to Y$ in $s\mathscr{C}$ is an acyclic fibration iff Nf is surjective inducing isomorphisms on homology, and that f is a cofibration iff Nf is injective with cokernel having projective objects of $\mathscr{C}$ in each dimenson. Therefore a cofibrant resolution of X in the sense described above is the same as a projective resolution of X in the usual sense, so the derived functors just defined coincide with those of Dold-Puppe [7].

2. **Homology and cohomology for universal algebras.** Let X be an object in a category $\mathscr{C}$ closed under finite projective limits and having sufficiently many projective objects. Let M be an abelian group object of the category $\mathscr{C}/X$ of objects over X and let P be a cofibrant simplicial resolution of X as in §1. We define the *cohomology groups* of X *with values in* M by the formula

$$D^q(X, M) = H^q\{\mathrm{Hom}_{\mathscr{C}/X}\,(P, M)\}, \tag{2.1}$$

where H^q denotes the homology of a cosimplicial abelian group with respect to the differential $\delta = \Sigma\,(-1)^i\delta_i$. By 1.3 the cohomology is independent of the choice of P and depends functorially on the pair (X, M), where a morphism $(X, M) \to (X', M')$ is a pair consisting of a morphism $X' \to X$ in $\mathscr{C}$ and a morphism $X' \times_X M \to M'$ of abelian group objects over X'.

Suppose now that $\mathscr{C}$ is an *algebraic category* by which we mean a category closed under inductive limits and having a set of small projective generators. For example if $\mathscr{C}$ has a single small projective generator, $\mathscr{C}$ is equivalent to the category of universal algebras defined by a set of operations and relations **[15]**. Then it may be shown that for any object X of $\mathscr{C}$, the category $(\mathscr{C}/X)_{ab}$ of abelian group objects over X is an abelian algebraic category and that moreover the abelianization functor $\mathrm{Ab}:\mathscr{C}/X \to (\mathscr{C}/X)_{ab}$, left adjoint to the forgetful functor, exists. Choosing a simplicial cofibrant resolution P of X, we define a simplicial object of $(\mathscr{C}/X)_{ab}$ by the formula

$$\mathrm{LAb}(X) = \mathrm{Ab}(P). \tag{2.2}$$

LAb (X) depends up to homotopy equivalence only on X and therefore gives rise via the normalization functor to an object of the derived category of $(\mathscr{C}/X)_{ab}$. LAb (X) should be thought of as being analogous to the complex of chains of a space X since one may calculate the cohomology of X with values in M by the formula

$$D^q(X, M) = H^q\{\mathrm{Hom}_{(\mathscr{C}/X)_{ab}}(\mathrm{LAb}\,(X), M)\}. \tag{2.3}$$

We define the qth *homology* object of X to be $D_q(X) = H_q\{\mathrm{LAb}\,(X)\}$.

The following proposition summarizes the properties of homology and cohomology in this general setting.

PROPOSITION 2.4. (i) *$D^*(X, M)$ is a cohomological functor of the object M of $(\mathscr{C}/X)_{ab}$.*

(ii) *There is a universal coefficient spectral sequence*

$$E_2^{pq} = \mathrm{Ext}^p_{(\mathscr{C}/X)_{ab}}(D_q(X), M) \Rightarrow D^{p+q}(X, M).$$

(iii) *If X is a projective object of $\mathscr{C}$, then $D_q(X) = D^q(X, M) = 0$ for all $q > 0$ and abelian group objects M over X.*

(iv) $D^0(X, M) = \mathrm{Hom}_{\mathscr{C}/X}(X, M)$, $D^1(X, M) =$ *isomorphism classes of objects Y over X which are torsors for M, i.e. endowed with a right action $Y \times_X M \to Y$ of M such that there exists an effective epimorphism $Z \to X$ such that $Z \times_X Y$ is isomorphic to $Z \times_X M$ with its natural right $Z \times_X M$ action.*

(v) *If X_i, $i \in I$ is a filtered inductive system of objects of $\mathscr{C}$, then* $\varinjlim D_q(X_i) \simeq D_q(\varinjlim X_i)$.

If P_i, $i \in I$ is a set of small projective generators for $\mathscr{C}$, then one obtains a simplicial cofibrant resolution of any object of $\mathscr{C}$ by taking the standard construction relative to the functor $X \to \prod_i \mathrm{Hom}_{\mathscr{C}}(P_i, X)$ from $\mathscr{C}$ to Sets$/I$ and its left adjoint. Thus the cohomology groups (2.1) are a kind of cotriple cohomology **[5]** and they coincide with the ones defined by André **[1]** using the set of P_i for models.

The cohomology groups $D^q(X, M)$ are also a special case of a very general definition of cohomology due to Grothendieck **[26]** as follows. The class of effective epimorphisms of $\mathscr{C}$ is stable under composition and base change and thus we obtain a pretopology in $\mathscr{C}$ by defining a covering of X to be a family consisting of a single effective epimorphism $Y \to X$. Representable functors are sheaves for the associated topology, so if M is an abelian group object over X, the functor $h_M = \mathrm{Hom}_{\mathscr{C}/X}(?, M)$ is a sheaf on $\mathscr{C}/X$ with the induced topology and sheaf cohomology groups $H^q(\mathscr{C}/X, h_M)$ are defined. A proof that $D^q(X, M) \simeq H^q(\mathscr{C}/X, h_M)$ is given in **[18, II, §5]**; alternatively it follows from a general result of Verdier on the calculation of sheaf cohomology by a modified Cech procedure **[26,** Appendix]. The advantage of the Grothendieck definition is that it applies even when there are not enough projectives in $\mathscr{C}$, e.g. categories of sheaves of algebras, provided the effective epimorphisms are closed under base change. When this last condition fails one may still consider other topologies e.g. (see **[12,** Exposé IV, 3.4]).

3. **Associative algebra cohomology.** Let A be a commutative ring with identity and let $\mathscr{C} = A$-Ass be the category of associative not necessarily commutative A-algebras with identity. In this section we will compute the cohomology of an object B of $\mathscr{C}$ in terms of Ext functors.

Let M be a B-bimodule, i.e. a left $B \otimes_A B^0$-module where B^0 is the opposed algebra to B, and let $B \oplus M$ denote the semidirect product A-algebra with multiplication $(b_1 \oplus m_1)(b_2 \oplus m_2) = b_1b_2 \ominus (b_1m_2 + m_1b_2)$. Then if C is an A-algebra over B we have canonical isomorphisms

$$(3.1)\qquad \begin{aligned} \mathrm{Hom}_{\mathscr{C}/B}(C, B \oplus M) &\cong \mathrm{Der}_A(C, M) \\ &\cong \mathrm{Hom}_{C\otimes_A C^0}(D_{C/A}, M) \\ &\cong \mathrm{Hom}_{B\otimes_A B^0}((B \otimes_A B^0) \otimes_{(C\otimes_A C^0)} D_{C/A}, M) \end{aligned}$$

where $\mathrm{Der}_A(C, M)$ is the abelian group of derivations of the A-algebra C with values in M regarded as a C-bimodule via the structural map $C \to B$ of C, and where $D_{C/A}$ is the C-bimodule of differentials of the A-algebra C. There is a canonical A-algebra derivation $d: C \to D_{C/A}$ which is universal for all derivations; one may prove that there is an exact sequence of C-bimodules

$$(3.2)\qquad 0 \longrightarrow D_{C/A} \xrightarrow{i} C \otimes_A C^0 \xrightarrow{\mu} C \longrightarrow 0$$

where $\mu(x \otimes y) = xy$ and id $(x) = 1 \otimes x - x \otimes 1$.

As $C \mapsto \mathrm{Der}_A(C, M)$ is a functor from $\mathscr{C}/B$ to abelian groups, it follows from 3.1 that $B \oplus M$ is naturally an abelian group object of $\mathscr{C}/B$. Conversely one shows easily that any abelian group object (more generally any monoid object) X of $\mathscr{C}/B$ is isomorphic to $B \oplus M$ with $M =$ kernel of structural map $X \to B$. Moreover in this way one obtains an equivalence of $(\mathscr{C}/B)_{ab}$ with the category of B-bimodules. Identifying these categories, 3.1 shows that the abelianization functor $\mathrm{Ab}: \mathscr{C}/B \to (\mathscr{C}/B)_{ab}$ is given by $C \mapsto (B \otimes_A B^0) \otimes_{(C\otimes_A C^0)} D_{C/A}$, hence

$$(3.3)\qquad D^q_{A\text{-Ass}}(B, M) \cong H^q\{\mathrm{Hom}_{P\otimes_A P^0}(D_{P/A}, M)\}$$

$$(3.4)\qquad \mathrm{LAb}(B) \cong (B \otimes_A B^0) \otimes_{(P\otimes_A P^0)} D_{P/A},$$

where P is a simplicial projective A-algebra resolution of B.

In order to put 3.3 in a more agreeable form, one shows that the right-hand side (since $D_{P/A}$ is a free P-bimodule) is isomorphic to the group $\mathrm{Ext}_{P\otimes_A P^0}{}^q(D_{P/A}, M)$ of morphisms from $D_{P/A}$ to the qth suspension $\Sigma^q M$ of M in the homotopy category of simplicial $P \otimes_A P^0$-modules [**18**, II, §6]. From the exact sequence 3.2, with C replaced by P, one obtains a long exact sequence yielding isomorphisms

$$D^q_{A\text{-Ass}}(B, M) \cong \mathrm{Ext}_{P\otimes_A P^0}{}^{q+1}(B, M) \quad \text{if } q > 0. \tag{3.5}$$

Since $P \to B$ induces isomorphisms on homology the right-hand side is isomorphic to $\mathrm{Ext}_{P\otimes_A P^0}{}^{q+1}(B, M)$. Now the homotopy category $\mathrm{Ho}\,(\mathscr{M}_R)$ of simplicial modules over a simplicial ring R depends only on the weak homotopy type of R in the sense that if $R \to R'$ is a map of simplicial rings inducing isomorphisms on homology, then the adjoint functors of extension and restriction of scalars induce an equivalence of $\mathrm{Ho}\,(\mathscr{M}_R)$ and $\mathrm{Ho}\,(\mathscr{M}_{R'})$. Consequently $\mathrm{Ext}_{P\otimes_A P^0}{}^*\,(B, M) \cong \mathrm{Ext}_{B\overset{\mathbf{L}}{\otimes}_A B^0}{}^*\,(B, M)$, where $B \overset{\mathbf{L}}{\otimes}_A B^0$ denotes any simplicial ring of the same weak homotopy type over $B \otimes_A B^0$ as $P \otimes_A P^0$. In particular if $\mathrm{Tor}^A_q\,(B, B) = 0$ for $q > 0$, then we may take $B \overset{\mathbf{L}}{\otimes}_A B^0$ to be $B \otimes_A B$, whence the category $\mathrm{Ho}\,(\mathscr{M}_{B\otimes_A B^0})$ is by Dold-Puppe just the full subcategory of the derived category of B-bimodules consisting of chain complexes, and hence $\mathrm{Ext}_{P\otimes_A P^0}{}^*(B, M)$ is isomorphic to the usual $\mathrm{Ext}_{B\otimes_A B^0}{}^*(B, M)$ of homological algebra. Thus combining 3.5 and 2.4 (iv), we have

PROPOSITION 3.6. *If B is an associative A-algebra and M is a B-bimodule, then*

$$\begin{aligned} D^q_{A\text{-Ass}}(B, M) &\cong \mathrm{Der}_A\,(B, M) && \text{if } q = 0,\\ &\cong \mathrm{Ext}_{B\overset{\mathbf{L}}{\otimes}_A B^0}{}^{q+1}(B, M) && \text{if } q > 0, \end{aligned}$$

where we may take $B \overset{\mathbf{L}}{\otimes}_A B \cong B \otimes_A B$ if $\mathrm{Tor}^A_q\,(B, B) = 0$ for $q > 0$.

Therefore if A is a field the associative algebra cohomology $D^*_{A\text{-Ass}}$ is with minor alterations the same as the Hochschild cohomology. This result should be compared with Barr [3], which indicates that the groups $\mathrm{Ext}_{B\overset{\mathbf{L}}{\otimes}_A B^0}{}^*(B, M)$ may be calculated by the differential graded algebra constructed by Shukla [**23**].

By essentially the same method used to prove 3·6 one may prove the following, where A-Lie is the category of Lie algebras over A, and where one identifies $(A\text{-Lie}/\mathfrak{J})_{ab}$ with the category of $\mathfrak{J}$-modules.

PROPOSITION 3.7. *If $\mathfrak{J}$ is a Lie algebra over A and M is a $\mathfrak{J}$-module, then*

$$\begin{aligned} D^q_{A\text{-Lie}}(\mathfrak{J}, M) &= \mathrm{Der}_A\,(\mathfrak{J}, M) && \text{if } q = 0,\\ &= \mathrm{Ext}^{q+1}_{U(P)}\,(A, M) && \text{if } q > 0, \end{aligned}$$

where P is a simplicial cofibrant A-Lie algebra resolution of $\mathfrak{J}$ and where U is the universal enveloping algebra functor. Furthermore if $\mathfrak{J}$ is flat as an A-module, then $U(P)$ may be replaced by $U(\mathfrak{J})$.

4. **Cohomology of commutative rings, the relative cotangent complex.** For the rest of this paper all rings shall be understood to be commutative with unit. Let

A be a ring, let $\mathscr{C}$ be the category of A-algebras and let B be an A-algebra. If M is a B-module let $B \oplus M$ be the trivial A-algebra extension of B by M given by the formulas $(b_1 \oplus m_1)(b_2 \oplus m_2) = b_1 b_2 \oplus (b_1 m_2 + b_2 m_1), a(b \oplus m) = ab \oplus am$. Then there are canonical isomorphisms

$$\text{(4.1)} \qquad \operatorname{Hom}_{\mathscr{C}/B}(C, B \oplus M) \cong \operatorname{Der}_A(C, M) \cong \operatorname{Hom}_B(B \otimes_C \Omega_{C/A}, M)$$

where $\operatorname{Der}_A(C, M)$ is the B-module of derivations of the A-algebra C with values in M, and where $\Omega_{C/A} = J/J^2, J = \operatorname{Ker}\{C \otimes_A C \to C\}$, is the C-module of Kähler differentials of the A-algebra C. The first formula shows that $B \oplus M$ is an abelian group object of $\mathscr{C}/B$, in fact a B-module object. One shows easily that the functor $M \mapsto B \oplus M$ is an equivalence of the category of B-modules with $(\mathscr{C}/B)_{ab}$. Identifying these two categories, the second part of 4.1 shows that the abelianization functor $\operatorname{Ab}: \mathscr{C}/B \to (\mathscr{C}/B)_{ab}$ is given by $\mathscr{C} \mapsto B \otimes_A \Omega_{C/A}$.

In accordance with §2 the *cohomology of the A-algebra B with values in the B-module M* is given by

$$\text{(4.2)} \qquad D^q(B/A, M) = H^q\{\operatorname{Der}_A(P, M)\}$$

where P is a simplicial cofibrant A-algebra resolution of B and where the expression on the right is the cohomology of the cosimplicial B-module $k \mapsto \operatorname{Der}_A(P_k, M)$. $D^*(B/A, M)$ is a cohomological functor of M and in low dimensions is, after interpreting 2.4(iv) given with the notation of the introduction by

$$\text{(4.3)} \qquad \begin{aligned} D^0(B/A, M) &= \operatorname{Der}_A(B, M) \\ D^1(B/A, M) &= \operatorname{Exalcomm}_A(B, M). \end{aligned}$$

The homology B-modules $D_q(B/A)$ of §2 are defined to be the homology of the simplicial B-module

$$\text{(4.4)} \qquad \operatorname{LAb}(B) = B \otimes_P \Omega_{P/A}.$$

We shall denote this simplical B-module by $\mathbf{L}_{B/A}$ in the following and call it the *cotangent complex* of the A-algebra B or of B relative to A. It is a cofibrant simplicial B-module, whose normalization is a chain complex of projective B-modules independent up to homotopy equivalence of the choice of P, and which therefore represents an object unique up to isomorphism of the derived category of the category of B-modules. Combining 4.1 and 4.2 we have

$$\text{(4.5)} \qquad D^q(B/A, M) = H^q\{\operatorname{Hom}_B(\mathbf{L}_{B/A}, M)\}.$$

Using the tensor product operation on B-modules, it is possible to introduce the *homology of the A-algebra B with values in M*

$$\text{(4.6)} \qquad D_q(B/A, M) = H_q(\mathbf{L}_{B/A} \otimes_B M)$$

related to the homology $D_q(B/A) = D_q(B/A, B)$ by a universal coefficient spectral sequence

$$\text{(4.7)} \qquad E^2_{pq} = \operatorname{Tor}^B_p(D_q(B/A), M) \Rightarrow D_{p+q}(B/A, M).$$

However, one sees from 4.5 and 4.6 that the cotangent complex is a finer invariant than either the cohomology or homology functors, and is therefore the basic object of study. This point of view is due to Grothendieck [9].

Suppose now that

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \tag{4.8}$$

is a commutative square of rings and that P (resp. P') is a cofibrant simplicial A- (resp. A'-) algebra resolution of B (resp. B'). By 1.3 there is a morphism $P \to P'$ of simplicial A-algebras over B' unique up to homotopy, hence a morphism of B' modules $B' \otimes_B (B \otimes_P \Omega_{P/A}) \to B' \otimes_{P'} \Omega_{P'/A'}$. Thus there is a morphism of simplicial B'-modules

$$B' \otimes_B \mathbf{L}_{B/A} \to \mathbf{L}_{B'/A'} \tag{4.9}$$

unique up to homotopy associated to 4.8 and consequently for any B'-module M' restriction homomorphisms

$$\begin{aligned} D^q(B'/A', M') &\to D^q(B/A, M') \\ D_q(B/A, M') &\to D_q(B'/A', M'). \end{aligned} \tag{4.10}$$

If $\{A_i \to B_i, i \in I\}$ is a filtered inductive system of morphisms of rings, then for each i let P_i be the cofibrant A_i-algebra resolution of B_i obtained using the standard construction associated to the pair of adjoint functors: "underlying set of an A_i-algebra" and "free A_i-algebra generated by the set." Using these resolutions to calculate the cotangent complex, one obtains the formulas

$$\begin{aligned} \mathbf{L}_{\lim B_i/\lim A_i} &= \lim \mathbf{L}_{B_i/A_i} \\ D_q(\lim B_i/\lim A_i, \lim M_i) &= \lim D_q(B_i/A_i, M_i) \end{aligned} \tag{4.11}$$

where $i \to M_i$ is a module for the functor $i \mapsto B_i$.

By analyzing the construction of a cofibrant A-algebra resolution P of B, one sees that if A is noetherian and B is an A-algebra of finite type, then P may be chosen so that P_k is a polynomial ring with finitely many variables for each $k \geq 0$. Using this P to calculate the cotangent complex one obtains

PROPOSITION 4.12. *If A is noetherian and B is an A-algebra of finite type, then $\mathbf{L}_{B/A}$ is isomorphic in the derived category of B-modules to a complex which is a free finite type B-module in each dimension. Consequently if M is a B-module of finite type, then for each q, $D^q(B/A, M)$ and $D_q(B/A, M)$ are B-modules of finite type.*

5. **The transitivity, flat base change, and vanishing theorems.** In this section we give the basic properties of the cohomology of commutative rings signaled in the introduction. These will be stated also as properties of the relative cotangent complex $\mathbf{L}_{B/A}$, which is identified via the normalization functor with an object of the

derived category of B-modules. We recall that the derived category [25] is endowed with a "suspension" or "translation" functor, here denoted by Σ, and that any exact sequence of complexes gives, via an analogue of the Puppe sequence construction in algebraic topology, a "distinguished triangle" in the derived category.

THEOREM 5.1. (*Transitivity*) *If $A \to B \to C$ are morphisms of rings, then there is a canonical distinguished triangle in the derived category of C-modules*

$$C \otimes_B \mathbf{L}_{B/A} \longrightarrow \mathbf{L}_{C/A} \longrightarrow \mathbf{L}_{C/B} \xrightarrow{\delta} \Sigma\,\{C \otimes_B \mathbf{L}_{B/A}\}.$$

Consequently if M is a C-module, there are exact sequences

$$0 \longrightarrow D^0(C/B, M) \longrightarrow D^0(C/A, M) \longrightarrow D^0(B/A, M) \xrightarrow{\delta} D^1(C/B, M) \longrightarrow \cdots.$$

The proof uses in an essential way the possibility of calculating the cotangent complex using any cofibrant resolution. Thus let P be a cofibrant simplicial A-algebra resolution of B and let Q be a cofibrant simplicial P-algebra resolution of C, i.e. $P \to Q \to C$ is a factorization into a cofibration followed by an acyclic fibration of the composition $P \to B \to C$. Then we have a diagram of simplicial rings

$$\begin{tikzcd}
& & & Q \arrow[d] \arrow[ddr] & \\
& P \arrow[urr] \arrow[dr] & & Q \otimes_P B \arrow[dr] & \\
A \arrow[ur] \arrow[rr] & & B \arrow[ur] \arrow[rr] & & C
\end{tikzcd}$$

and an exact sequence of C-modules

$$0 \to C \otimes_B (B \otimes_P \Omega_{P/A}) \to C \otimes_Q \Omega_{Q/A} \to C \otimes_{Q \otimes_P B} \Omega_{Q \otimes_P B/B} \to 0.$$

The first and second term may be identified with $C \otimes_B \mathbf{L}_{B/A}$ and $\mathbf{L}_{C/A}$ respectively, the distinguished triangle associated to this exact sequence is the desired one in 5.1 once we identify the last term with $\mathbf{L}_{C/B}$. For this it suffices to show that $Q \otimes_P B$ is a simplicial cofibrant B-algebra resolution of C. $B \to Q \otimes_P B$ is a cofibration since it is the base change of the cofibration $P \to Q$. To see that $Q \otimes_P B \to C$ induces isomorphisms on homology, it suffices to do the same for the map $Q \to Q \otimes_P B$, which results by applying the functor $Q \otimes_P ?$ to the acyclic fibration $P \to B$. Now the homological properties of the tensor product may be analyzed in terms of Künneth spectral sequences [18, II, 6]

$$\text{(5.2)} \qquad \begin{aligned} E^2_{pq} &= H_p\{\operatorname{Tor}^R_q(X, Y)\} \Rightarrow H_{p+q}(X \overset{\mathbf{L}}{\otimes}_R Y) \\ E^2_{pq} &= \operatorname{Tor}^{H(R)}_p(H(X), H(Y))_q \Rightarrow H_{p+q}(X \overset{\mathbf{L}}{\otimes}_R Y) \end{aligned}$$

where X (resp. Y) is a left (resp. right) simplicial module over a not necessarily commutative simplicial ring R. Applying these we get a diagram

$$\begin{tikzcd} H_p\{\mathrm{Tor}_q^P(Q,P)\} \Rightarrow H_{p+q}(Q \overset{\mathbf{L}}{\otimes}_P P) \arrow[d] & \Leftarrow \mathrm{Tor}_p^{H(P)}(H(Q),H(P))_q \arrow[d, "\cong"] \\ H_p\{\mathrm{Tor}_q^P(Q,B)\} \Rightarrow H_{p+q}(Q \overset{\mathbf{L}}{\otimes}_P B) & \Leftarrow \mathrm{Tor}_p^{H(P)}(H(Q),H(B))_q. \end{tikzcd}$$

As $P \to Q$ is a cofibration, Q in each dimension is a retract of a free P-algebra, hence Q is projective as P-module and the Tor's on the left vanish for $q > 0$. Thus the spectral sequences on the left degenerate showing that $H_*(Q) \xrightarrow{\sim} H_*(Q \otimes_P B)$ and concluding the proof of the theorem.

THEOREM 5.3. (*Flat base change.*) *If B and C are A-algebras such that* $\mathrm{Tor}_q^A(B, C) = 0$ *for* $q > 0$, *then there are isomorphisms in the derived category of* $B \otimes_A C$-*modules*

$$\mathbf{L}_{B\otimes_A C/C} \cong B \otimes_A \mathbf{L}_{C/A}$$

$$\mathbf{L}_{B\otimes_A C/A} \cong (\mathbf{L}_{B/A} \otimes_A C) \oplus (B \otimes_A \mathbf{L}_{C/A}).$$

Consequently if M is a $B \otimes_A C$-module, there are isomorphisms

$$D^q(B \otimes_A C/C, M) \cong D^q(B/A, M)$$

$$D^q(B \otimes_A C/A, M) \cong D^q(B/A, M) \oplus D^q(C/A, M).$$

If P is a cofibrant A-algebra resolution of B and Q is a cofibrant A-algebra resolution of C, then using the Künneth spectral sequences 5.2 and the fact that P and Q are projective as A-modules, one sees that $B \otimes_A Q$ (resp. $P \otimes_A Q$) is a cofibrant B-(resp. A-) algebra resolution of $B \otimes_A C$. Using these resolutions one obtains the formulas of the theorem.

The following vanishing properties of the cotangent complex may be deduced with the aid of the spectral sequence of §6. However, André has shown by very beautiful arguments that they follow directly from the transitivity and flat base change theorems, so we state them here. Recall that a chain complex K of B-modules is said to have projective (resp. Tor) dimension $\leq n$ if $H^q\{\mathrm{Hom}_B(K, M)\} = 0$ (resp. $H_q(K \otimes_B M) = 0$) for $q > n$ and all B-modules M.

THEOREM 5.4. (i) *If S is a multiplicative system in A, then* $\mathbf{L}_{S^{-1}A/A} = 0$.

(ii) *If* Spec $B \to$ Spec A *is étale* **[11,** IV], *then* $\mathbf{L}_{B/A} = 0$

(iii) *If* Spec $B \to$ Spec A *is smooth* **[11,** IV], *then* $\mathbf{L}_{B/A}$ *is isomorphic in the derived category to* $\Omega_{B/A}$ *and has projective dimension* 0.

(iv) *If* Spec $B \to$ Spec A *is a local complete intersection morphism* **[10]**, *then* $\mathbf{L}_{B/A}$ *has projective dimension* ≤ 1.

The assertions (ii)-(iv) admit the following partial converses.

THEOREM 5.5. *If A is noetherian and B is an A-algebra of finite type, then*

(i) Spec $B \to$ Spec A *is smooth (resp. étale)* $\Leftrightarrow D^q(B/A, M) = 0$ *for all B-modules M of finite type and* $q = 1$ (*resp.* $q = 0, 1$).

(ii) Spec $B \to$ Spec A *is a local complete intersection morphism* $\Leftrightarrow D^2(B/A, M) = 0$ *for all B-modules M of finite type.*

5.5 (i) is Grothendieck's infinitesimal criterion for smoothness [**11**, IV, §17], (ii) is due to Lichtenbaum-Schlessinger [**16**].

Unsolved Problem. Characterize the morphisms $A \to B$, where A is noetherian and B is an A-algebra of finite type, for which the cotangent complex is of finite projective dimension. What computations we have been able to make show that this is rare and support the following conjectures:

Conjecture 5.6. If $\mathbf{L}_{B/A}$ is of finite projective dimension then it is of projective dimension ≤ 2.

Conjecture 5.7. If $\mathbf{L}_{B/A}$ is of finite projective dimension and if B is of finite Tor dimension as an A-module, then Spec $B \to$ Spec A is a local complete intersection morphism.

6. **The fundamental spectral sequence.** In this section we retain the notations of the preceding two sections except that certain algebras of homology such as $\mathrm{Tor}_*^A(B, B)$ are not commutative but rather anticommutative with respect to the grading. With this exception which will always be clear from the context, rings are understood to be commutative.

If $A \to B$ is a morphism of rings and M is a B-module, and if we express B as a quotient of a polynomial ring P over A, then the transitivity exact sequence 5.1 applied to $A \to P \to B$ combined with the fact that cohomology vanishes on projectives (2.4, *iii*) yields isomorphisms

$$D^q(B/P, M) \xrightarrow{\sim} D^q(B/A, M) \qquad q \geq 2. \tag{6.1}$$

This shows that the calculation of the cohomology in dimensions ≥ 2 reduces to the case in which $B = A/I$ where I is an ideal in A. In this case there is a spectral sequence relating the cotangent complex $\mathbf{L}_{B/A}$ to the algebra $\mathrm{Tor}_*^A(B, B)$ which may be used to obtain information about the former.

If R is a simplicial ring and X is a simplicial R-module, and if F is a functor on the category of pairs (B, M) consisting of a ring and a module with dihomomorphisms for morphisms in the category, then by $F(R, X)$ we denote the simplicial object obtained by applying F dimension-wise. Examples of such F are the symmetric and exterior algebra functors

$$\begin{aligned} S^B M &= \bigoplus_{n=0}^{\infty} S_n^B M \\ \Lambda^B M &= \bigoplus_{n=0}^{\infty} \Lambda_n^B M \end{aligned} \tag{6.2}$$

where the latter is anticommutative.

THEOREM 6.3. *If $A \to B$ is a morphism of rings such that $B \otimes_A B \xrightarrow{\sim} B$ (for example if $B = S^{-1}(A/I)$, I an ideal in A, S a multiplicative system in A/I), then there* *is a first quadrant homological spectral sequence*

$$E^2_{pq} = H_{p+q}\{S^B_q \mathbf{L}_{B/A}\} \Rightarrow \mathrm{Tor}^A_{p+q}(B, B)$$

of bigraded algebras, anticommutative for the total degree $p + q$. *This spectral sequence has the following properties:*

Low dimensional isomorphisms:

$$D_0(B/A) = 0$$
$$D_1(B/A) \cong \mathrm{Tor}^A_1(B, B) \tag{6.4}$$

Edge homomorphisms:

$$\mathrm{Tor}^A_n(B, B) \to D_n(B/A) \tag{6.5}$$

$$\Lambda^B_n D_1(B/A) \to \mathrm{Tor}^A_n(B, B), \tag{6.6}$$

where 6.5 *annihilates the decomposable elements of the algebra* $\mathrm{Tor}^A_*(B, B)$ *and where* 6.6 *is the unique anticommutative B-algebra morphism extending the isomorphism* 6.4.

Five term exact sequence:

$$\mathrm{Tor}^A_3(B, B) \to D_3(B/A) \xrightarrow{d_2} \Lambda^B_2 D_1(B/A) \to \mathrm{Tor}^A_2(B, B) \to D_2(B/A) \to 0. \tag{6.7}$$

THEOREM 6.8. *If* $B \otimes_A B \xrightarrow{\sim} B$ *and* M *is a B-module, then there is a first quadrant cohomological spectral sequence of B-modules*

$$E_2^{pq} = H^{p+q}\{\mathrm{Hom}_B(S^B_q \mathbf{L}_{B/A}, M)\} \Rightarrow \mathrm{Ext}^{p+q}_A(B, M).$$

The spectral sequence of Theorem 6.3 is an analogue for simplicial rings of the lower central series spectral sequence for simplicial groups of Curtis [**6**] and is derived in the following way. Let P be a simplicial cofibrant A-algebra resolution of B and let $Q = P \otimes_A B$, $J = \mathrm{Ker}\{P \otimes_A B \to B\}$. Filtering Q by the powers of the simplicial ideal J, one obtains a (possibly nonconvergent) spectral sequence

$$E^2_{pq} = H_{p+q}(J^q/J^{q+1}) \Rightarrow H_{p+q}(Q) \cong \mathrm{Tor}^A_{p+q}(B, B), \tag{6.9}$$

which converges provided the connectivity of J^q goes to infinity with q. Now by definition $J/J^2 \cong B \otimes_P \Omega_{P/A} = \mathbf{L}_{B/A}$, and moreover $J^q/J^{q+1} \simeq S^B_q(J/J^2)$; in effect P may be taken to be a polynomial ring in each dimension, whence Q is a polynomial ring and J is the ideal generated by the variables. Thus 6.9 is the spectral sequence of Theorem 6.3. The convergence of the spectral sequence follows from the formula $H_q(J^k) = 0$ if $q < k$, which may be proved either by the arguments of Curtis [**6**, §4] or by using Theorem 6.12 below, which applies since $B \otimes_A B \xrightarrow{\sim} B$ implies that $H_0(J) = 0$ and since for each k, J_k is a regular ideal of Q_k in the sense of the following definition.

DEFINITION 6.10. An ideal I in a ring A will be called *quasiregular* (resp. *regular*) if I/I^2 is a flat (resp. projective) A/I-module and if the canonical morphism of anticommutative algebras

$$\Lambda_*^{A/I}(I/I^2) \to \mathrm{Tor}_*^A(A/I, A/I)$$

is an isomorphism.

It may be proved that if I is quasiregular, then the canonical algebra morphism

$$S_*^{A/I}(I/I^2) \to \bigoplus I^n/I^{n+1} \tag{6.11}$$

is an isomorphism. Furthermore if A is noetherian then I is regular iff it is quasi-regular iff it is generated locally by an A-regular sequence. For nonnoetherian rings in general the terminology used here differs from that of [**11**, IV], but seems better adapted for the cohomology theory of commutative rings. For example we have the following connectivity result.

THEOREM 6.12. *Let J be a simplicial ideal in a simplicial ring Q such that J_k is quasiregular in Q_k for each $k \geq 0$. If $H_0(J) = 0$, then $H_k(J^n) = 0$ for $k < n$.*

We now give some applications of the spectral sequence 6.3. If M is a B-module and q is an integer, we denote by $M[q]$ the object of the derived category of B-modules determined up to isomorphism by the condition that its homology groups are M in degree q and zero elsewhere. Here is a characterization of regular and quasiregular ideals using the cotangent complex.

THEOREM 6.13. *Suppose $B = A/I$. Then the following conditions are equivalent:*
(i) *I is a quasiregular ideal of A*
(ii) *$D_q(B/A, M) = 0$ for all $q \geq 2$ and all B-modules M*
(iii) *I/I^2 is a flat B-module and $\mathbf{L}_{B/A} \cong I/I^2[1]$.*

COROLLARY 6.14. *If $B = A/I$, the following are equivalent:*
(i) *I is a regular ideal of A*
(ii) *$D^q(B/A, M) = 0$ for all $q \geq 2$ and all B-modules M*
(iii) *I/I^2 is a projective B-module and $\mathbf{L}_{B/A} \cong I/I^2\,[1]$.*

Another application of the spectral sequence is the following Artin-Rees type theorem and its corollaries which was suggested by the fact that $D^*(B/A, M)$ is the sheaf cohomology of a site (§2).

THEOREM 6.15. *Let A be a noetherian ring, let I be an ideal in A, and let $A_n = A/I^{n+1}$, $n \geq 0$. Then for each $q \geq 0$, there exists an N such that the natural homomorphism*

$$D_q(A_{n+N}/A, A_0) \to D_q(A_n/A, A_0)$$

is zero for all $n \geq 0$.

COROLLARY 6.16. *Let A be a noetherian ring, let B be the localization of an A-algebra of finite type C with respect to a multiplicative system in C, and let M be a B-module (not necessarily finitely generated). Then given $u \in D^q(B/A, M)$ with $q > 0$, there exists a surjective morphism $p: B' \to B$ of A-algebras, where the kernel of p is a finitely generated nilpotent ideal of B', such that $p^*u = 0$, where p^* is the natural homomorphism $D^q(B/A, M) \to D^q(B'/A, M)$.*

COROLLARY 6.17. *Suppose A is noetherian, B is an A-algebra of finite type and M is a B-module. Let T be the category of finite type A-algebras over B endowed with the topology* [**26**] *associated to the pretopology in which the covering families are single morphisms $B'' \to B'$ which are surjective and have nilpotent kernel. Then $D^q(B/A, M)$ is isomorphic to the cohomology of the sheaf $B' \mapsto \mathrm{Der}\,(B'/A, M)$ on the site T.*

For $q \geq 2$ the noetherian hypotheses of 6.15–6.17 are essential. 6.17 was originally proved in order to define as sheaf cohomology global groups $D^*(X/Y, F)$ generalizing the $D^*(B/A, M)$ for any finite type morphism $f: X \to Y$ of noetherian schemes and any quasi-coherent sheaf F on X. However Illusie [**14**] has recently found a general definition of the cotangent complex for any morphism of ring objects in a topos, rendering this method of little interest.

7. **Degeneracy of the fundamental spectral sequence in characteristic zero.** Suppose that B is an A-algebra such that $B \otimes_A B \xrightarrow{\sim} B$ and such that $\mathrm{Tor}_*^A\,(B, B)$ is flat as a B-module. Then [**2**] $\mathrm{Tor}_*^A\,(B, B)$ has a natural structure of a commutative Hopf algebra over B. In particular if B is of characteristic zero, that is, an algebra over the rational numbers, then there is a canonical Poincaré-Birkhoff-Witt isomorphism

$$\tilde{S}^B\{Q_*\} \simeq \mathrm{Tor}_*^A\,(B, B) \tag{7.1}$$

of graded anticommutative B-algebras, where Q_* is the indecomposable quotient B-module of $\mathrm{Tor}_*^A\,(B, B)$, and where $\tilde{S}^B = \oplus\, \tilde{S}_n^B$ is the anticommutative symmetric algebra functor on graded B-modules. This isomorphism is not in general compatible with the coalgebra structures.

Also when B is of characteristic zero, the functor $M \mapsto S_m^B M$ on the category of B-modules is a direct summand of the functor $M \mapsto M^{\otimes m}$. Consequently by the Künneth spectral sequences 5.2, the natural map

$$\tilde{S}_m^B\{H_*(K)\} \to H_*\{S_m^B K\} \tag{7.2}$$

is an isomorphism in dimensions $\leq q$ if K is a simplicial B-module such that K_n and $H_n(K)$ are flat B-modules for $n < q$. Using this fact and the isomorphism 7.1, one may analyze the fundamental spectral sequence and obtain the following results.

THEOREM 7.3. *Suppose that B is an A-algebra such that $B \otimes_A B \xrightarrow{\sim} B$. Assume that B is of characteristic zero and that $\mathrm{Tor}_*^A\,(B, B)$ is flat as a B-module. Then the spectral sequence* 6.3 *degenerates and yields an isomorphism of $D_*(B/A)$ with the indecomposable quotient B-module of $\mathrm{Tor}_*^A\,(B, B)$. Furthermore there is a canonical B-algebra isomorphism*

$$\tilde{S}^B\{D_*(B/A)\} \cong \mathrm{Tor}_*^A\,(B, B),$$

$D_*(B/A)$ is a flat B-module, and $D_*(B/A)$ has a natural (anticommutative) graded Lie coalgebra structure over B.

COROLLARY 7.4. *Let A be a local noetherian ring with residue field k of characteristic zero. Then $D^*(k/A, k)$ is isomorphic to the subspace of primitive elements of the cocommutative Hopf algebra* $\operatorname{Ext}_A^*(k, k)$. *In particular $D^*(k/A, k)$ has a natural structure as a (anticommutative) graded Lie algebra.*

COROLLARY 7.5. *Let A and B be local noetherian rings with the same residue field k of characteristic zero, and let $A \times_k B$ be their fibre product over k (geometrically* Spec $(A \times_k B)$ *is the wedge of* Spec A *and* Spec B). *Then*

$$D^*(k/A, k) \vee D^*(k/B, k) \xrightarrow{\sim} D^*(k/A \times_k B, k)$$

where $\vee$ denotes the direct sum in the category of (anticommutative) graded Lie algebras.

EXAMPLE 7.6. Let $A = k \oplus m$ where $m^2 = 0$. Then $\operatorname{Ext}_A^*(k, k)$ is the tensor algebra over k generated by $\operatorname{Ext}_A^1(k, k) = m'$, so $D^*(k/A, k)$ is the free graded Lie algebra generated by m' in degree 1.

EXAMPLE 7.7. Let A, B be regular local rings of dimensions p and q respectively with common residue field k. Then $D^*(k/A \times_k B, k)$ is the graded *Lie* algebra with generators $x_1, \ldots, x_p, y_1, \ldots, y_q$ of degree 1 and subject to the relations $[x_i, x_j] = 0$ for $1 \leq i, j \leq p$, $[y_i, y_j] = 0$ for $1 \leq i, j \leq q$.

When $\operatorname{Tor}_*^A(B, B)$ fails to be flat over B, it is easy to find examples where the conclusions of 7.3 fail to hold, e.g. $B = A/I$ where A is a regular local ring and I is not a regular ideal. However we have the following degeneracy theorem proved by the techniques of §9.

THEOREM 7.8. *Suppose that A is an augmented B-algebra and that B is of characteristic zero. Then the spectral sequences* 6.3 *and* 6.8 *are degenerate; in fact there are canonical isomorphisms*

$$\bigoplus_{q=0}^{n} H_n\{S_q^B \mathbf{L}_{B/A}\} \cong \operatorname{Tor}_n^A(B, B)$$

$$\bigoplus_{q=0}^{n} H^n\{\operatorname{Hom}_B(S_q^B \mathbf{L}_{B/A}, M)\} \cong \operatorname{Ext}_A^n(B, M).$$

8. **Relations with associative algebra cohomology.** Let $A \to B$ be a morphism of (commutative) rings and let M be a B-module. If R is a simplicial A-algebra resolution of B which in every dimension is flat as an A-module, then by the Künneth spectral sequences 5.2. the weak homotopy type of the simplicial B-algebra $R \otimes_A B$ is independent of R. This means that if R' is another such resolution of B, then $R \otimes_A B$ and $R' \otimes_A B$ are canonically isomorphic in the homotopy category Ho (sA-alg) of simplicial A-algebras, which is the category obtained from the category of simplicial A-algebras by formally inverting the morphisms which induce isomorphisms on homology. We denote by $B \overset{\mathbf{L}}{\otimes}_A B$ the object $R \otimes_A B$, unique up to canonical isomorphism, of the category Ho (sA-alg). According to 3.6, the cohomology of B as just an associative A-algebra with values in M

may be calculated as certain Ext groups of morphisms in the homotopy category of simplicial modules over $B \overset{\mathbf{L}}{\otimes}_A B$. Therefore the following spectral sequences may be regarded as a relation between the cohomologies of B as a commutative A-algebra and as just an associative A-algebra.

THEOREM 8.1. *There is a spectral sequence of B-algebras*

$$E^2_{pq} = H_p\{\Lambda^B_q \mathbf{L}_{B/A}\} \Rightarrow \mathrm{Tor}_{p+q}{}^{B \overset{\mathbf{L}}{\otimes}_A B}(B, B) \tag{8.2}$$

anticommutative for the total degree $p + q$, where Λ is the exterior algebra functor. There is also a spectral sequence of B-modules

$$E_2^{pq} = H^p\{\mathrm{Hom}_B\,(\Lambda^B_q \mathbf{L}_{B/A}, M)\} \Rightarrow \mathrm{Ext}_{B \overset{\mathbf{L}}{\otimes}_A B}{}^{p+q}(B, M) \tag{8.3}$$

Furthermore if $\mathrm{Tor}^A_q\,(B, B) = 0$ *for* $q > 0$, *then* $B \overset{\mathbf{L}}{\otimes}_A B$ *may be replaced by* $B \otimes_A B$ *in these spectral sequences.*

We sketch the proof. If R is an augmented B-algebra, let $\mathscr{B}(R)$ be the total bar construction of R, and let $\bar{\mathscr{B}}(R) = B \otimes_R \mathscr{B}(R)$. Let P be a cofibrant simplicial A-algebra resolution of B so that $P \otimes_A B = B \overset{\mathbf{L}}{\otimes}_A B$. Let $\mathscr{B}(P \otimes_A B)$ be the result of applying the functor $\mathscr{B}$ dimension-wise to the simplicial augmented B-algebra $P \otimes_A B$. It is a chain complex of simplicial $P \otimes_A B$ modules. Let us think of the simplicial structure of $\mathscr{B}(P \otimes_A B)$ as being in the vertical direction and the differentials of the bar construction as running horizontally, and let $(N^h)^{-1}\mathscr{B}(P \otimes_A B)$ be the simplicial object in the category of simplicial $P \otimes_A B$ modules obtained by applying the inverse of the normalization functor in the horizontal direction. Then the diagonal $\Delta(N^h)^{-1}\mathscr{B}(P \otimes_A B)$ of this double simplicial object is a flat $P \otimes_A B$-module resolution of B, hence $\Delta(N^h)^{-1}\bar{\mathscr{B}}(P \otimes_A B)$ is isomorphic to $B \overset{\mathbf{L}}{\otimes}_{B \overset{\mathbf{L}}{\otimes}_A B} B$. The double simplicial object $(N^h)^{-1}\bar{\mathscr{B}}(P \otimes_A B)$ gives rise to a spectral sequence with abutment

$$H_n(B \overset{\mathbf{L}}{\otimes}_{B \overset{\mathbf{L}}{\otimes}_A B} B) = \mathrm{Tor}_n{}^{B \overset{\mathbf{L}}{\otimes}_A B}(B, B)$$

and whose E^2 term is

$$E^2_{pq} = H^v_p H^h_q \bar{\mathscr{B}}(P \otimes_A B) = H_p\,\mathrm{Tor}_q{}^{P \otimes_A B}(B, B) = H_p\{\Lambda^B_q \mathbf{L}_{B/A}\},$$

thus giving the spectral sequence 8.2. The other is derived by applying the functor $\mathrm{Hom}_B\,(?, M)$ to $\Delta(N^h)^{-1}\mathscr{B}(P \otimes_A B)$.

The bar construction $\bar{\mathscr{B}}(R)$ has the structure of a commutative differential graded Hopf algebra over B when R is a commutative augmented B-algebra, hence if B is of characteristic 0, there is a canonical Poincaré-Birkhoff-Witt isomorphism of differential graded algebras

$$\bar{\mathscr{B}}(R) \simeq \tilde{S}^B(Q), \tag{8.4}$$

where Q is the indecomposable part. Using this fact one can show that there is a canonical isomorphism

$$S^B\{\mathbf{L}_{B/A}\} \simeq B \overset{\mathbf{L}}{\otimes}_{B \overset{\mathbf{L}}{\otimes}_A B} B \tag{8.5}$$

in the homotopy category of simplicial B-algebras. In slightly more concrete terms this amounts to the following result.

THEOREM 8.6. *If B is of characteristic zero, then the spectral sequences of* 8.1 *are degenerate and in fact there are canonical isomorphisms*

$$\bigoplus_{p+q=n} H_p\{\Lambda^B_q \mathbf{L}_{B/A}\} \simeq \mathrm{Tor}_n{}^{B \overset{\mathbf{L}}{\otimes}_A B}(B, B)$$

$$\bigoplus_{p+q=n} H^p\{\mathrm{Hom}_B\,(\Lambda^B_q \mathbf{L}_{B/A}, M)\} \simeq \mathrm{Ext}_{B \overset{\mathbf{L}}{\otimes}_A B}{}^n(B, B).$$

COROLLARY 8.7. *When B is of characteristic zero, then commutative algebra cohomology $D^q(B/A, M)$ is canonically a direct summand of the associative algebra cohomology* $\mathrm{Ext}_{B \overset{\mathbf{L}}{\otimes}_A B}{}^{q+1}(B, M)$.

The corollary in the case where A is a field is due to Barr [**4**]. His proof uses an explicit formula for the projection operator of $\bar{\mathscr{B}}(R)$ onto Q in 8.4.

9. **Differential graded algebra resolutions in characteristic zero.** In this section we give an example of what appears to be a rather general phenomon, that in characteristic zero simplicial objects can be replaced by differential graded objects (see also [**20**]).

Let $\mathscr{D}$ be the abelian category of chain complexes of abelian groups. The tensor product of chain complexes is a coherently associative and commutative tensor product operation in the sense of MacLane [**17**], where the isomorphism of commutativity $T: X \otimes Y \simeq Y \otimes X$ is that of Koszul: $T(x \otimes y) = (-1)^{pq} y \otimes x$ if x and y have degrees p and q respectively. Much of the following discussion may be generalized to a class of such abelian categories with tensor product.

A ring R in $\mathscr{D}$ with respect to the tensor structure is just a differential graded (DG) ring in the usual sense. If M is an object of $\mathscr{D}$ which is a left R-module, i.e. a left DG module over R, and if N is a right R-module, then the tensor product $N \otimes_R M$ and its derived functors are also objects of $\mathscr{D}$. If R is a commutative ring in $\mathscr{D}$, that is, an anticommutative DG ring in the usual sense, and if M is an R-module, then the symmetric algebra $S^R\{M\} = \bigoplus_{q=0}^{\infty} S^R_q M$ is defined by the standard universal mapping property relative to commutative R-algebras. (When R is the DG ring which is the ring B concentrated in degree zero and M is a complex of B-modules, then this symmetric algebra is also denoted by $\tilde{S}$ in this paper.) We may also define the exterior algebra $\Lambda^R M = \bigoplus_{q=0}^{\infty} \Lambda^R_q M$ by means of the standard universal mapping property relative to graded anticommutative R-algebras in $\mathscr{D}$ (thus $\Lambda^R M = \oplus\,(\Lambda^R_q M)_n$ is a bigraded ring anticommutative for the degree $q + n$).

One may prove that if $A \to B$ is a morphism of commutative rings in $\mathscr{D}$, then $\mathrm{Tor}^A_*\,(B, B)$ is a graded anticommutative ring in $\mathscr{D}$. Hence if $B = A/I$, there is a canonical isomorphism of graded anticommutative B-algebras in $\mathscr{D}$

$$(9.3) \qquad \Lambda_*^B(I/I^2) \to \operatorname{Tor}_*^A(B, B).$$

We now carry over the definition 6.10 to this context and say that I is quasiregular if I/I^2 is B-flat and if 9.3 is an isomorphism.

PROPOSITION 9.4. *Suppose that B is a commutative ring in $\mathscr{D}$ and M is a B-module. Assume that M is B-flat and that $n: M \to M$ is an isomorphism for each integer $n \neq 0$. Then the augmentation ideal of $S^B M$ is quasiregular.*

In effect the Koszul complex $S^B M \otimes_B \Lambda_*^B M$ is a flat resolution of B as an $S^B M$-algebra and hence can be used to calculate the Tor in 9.3.

Let $u: A \to B$ be a morphism of (ordinary) commutative rings and regard u as a morphism of DG rings concentrated in degree zero. By a *DG anticommutative A-algebra resolution of B* we mean a factorization $u = pi$, $i: A \to R$, $p: R \to B$ in $\mathscr{D}$ such that p induces isomorphisms on homology.

THEOREM 9.5. *Suppose $A \to B$ is a morphism of commutative rings and that B is of characteristic zero. Let R be a DG anticommutative A-algebra resolution of B such that R is A-flat and such that the augmentation ideal J of the DG ring $R \otimes_A B$ is quasiregular. Then there is a canonical isomorphism $J/J^2 \simeq \mathbf{L}_{B/A}$ in the derived category of B-modules.*

If R is a free DG anticommutative A-algebra, i.e. ignoring differentials R is isomorphic to $\tilde{S}^A\{N\}$ where N is a free complex of A-modules, then the augmentation ideal of $R \otimes_A B$ is quasiregular because of 9.4 and the fact that Tor's in $\mathscr{D}$ do not depend on the differentials. Therefore Theorem 9.5 implies that the cotangent complex can be calculated by using free DG anticommutative A-algebra resolutions of B, provided B is of characteristic zero. One may show that this is equivalent to the method used by Lichtenbaum-Schlessinger **[16]** to define $D^q(B/A, M)$ for $q \leq 2$.

We now apply this theorem to the bar construction of an augmented B-algebra A where B is of characteristic zero. Let $\mathscr{B}(A)$ be the total bar construction and $\overline{\mathscr{B}}(A) = B \otimes_A \mathscr{B}(A)$. If A is B-flat, then $\mathscr{B}(A)$ is a flat DG anticommutative A-algebra resolution of B. Moreover $\overline{\mathscr{B}}(A)$ is a commutative DG Hopf algebra over B so by the Poincaré-Birkhoff-Witt isomorphism 8.4 it is isomorphic as a DG B-algebra to $\tilde{S}Q$, where Q is the indecomposable part. Using 9.4 it follows that the augmentation ideal of $\overline{\mathscr{B}}(A)$ is quasiregular. By the theorem the cotangent complex $\mathbf{L}_{B/A}$ is Q. Thus we have

COROLLARY 9.6. *If A is a flat augmented B-algebra and if B is of characteristic zero, then $\mathbf{L}_{B/A} \simeq J/J^2$ where J is the augmentation ideal in the bar construction $\overline{\mathscr{B}}(A)$. Therefore*

$$(9.7) \qquad \mathbf{L}_{B/A} \simeq L'(\bar{A}[1])$$

where $\bar{A}[1]$ *is the augmentation ideal of* A *situated in degree* 1, *where* L' *is the free anticommutative graded coLie algebra functor, and where the differential on the right hand side of* 9.7 *is the unique degree* -1 *coderivation which in dimension* 2 *is the map* $S_2^B \bar{A} \to \bar{A}$ *given by the multiplication of* A.

Suppose now that B is a flat A-algebra of characteristic zero and consider $B \otimes_A B$ as an augmented B-algebra where the maps $B \to B \otimes_A B \to B$ are given by $b \mapsto b \otimes 1$, $x \otimes y \mapsto xy$. Applying 9.6 one obtains the following result allowing one to calculate the commutative algebra cohomology by the cochain complex of Harrison [**13**].

COROLLARY 9.8. *If* B *is a flat* A*-algebra of characteristic zero, then there is an isomorphism* $\mathbf{L}_{B/A} \simeq \Sigma^{-1}(J/J^2)$ *in the derived category of* B*-modules, where* Σ *is the suspension functor, and where* J *is the augmentation ideal of the bar construction* $\bar{\mathscr{B}}(B \otimes_A B)$. *Moreover if* B *is projective as an* A*-module, then*

$$D^q(B/A, M) = H^{q+1}\{\mathrm{Harr}^*(B/A, M)\}, \qquad \textit{where} \tag{9.9}$$

$$\mathrm{Harr}^q(B/A, M) = \{f \in \mathrm{Hom}_A(\otimes_A^q B, M), f \textit{ vanishes on decomposable elements for the shuffle product}\}$$

$$\begin{aligned}(df)(b_1, \ldots, b_{q+1}) = {} & b_1 f(b_2, \ldots, b_{q+1}) \\ & + \sum_{i=1}^{q} (-1)^i f(b_1, \ldots, b_i b_{i+1}, \ldots, b_{q+1}) \\ & + (-1)^{q+1} b_{q+1} f(b_1, \ldots, b_q).\end{aligned}$$

When A is a field, the formula 9.9 is due to Barr [**4**].

10. **Local noetherian rings.** Let A be a local noetherian ring with residue field k and maximal ideal m_A. Suppose that A is the quotient of a regular local ring R, for example if A is complete. R may be chosen to be of minimal dimension, in which case $m_R/m_R^2 \xrightarrow{\sim} m_A/m_A^2 \simeq D_1(k/A)$. Moreover the minimal number of generators for the ideal $\mathrm{Ker}\{R \to A\}$ is then $\dim_k D_2(k/A)$. These results may be generalized as follows.

PROPOSITION 10.1. (i) *If* A *is a local ring which is the quotient of a regular local ring, then there exists a simplicial resolution* P *of* A *such that each* P_q *is a regular local ring and such that each simplicial operation* $P_q \to P_r$ *is a local homomorphism. Moreover there is a canonical isomorphism*

$$m_P/m_P^2 \simeq \Sigma^{-1}\,\mathbf{L}_{k/A} \tag{10.2}$$

in the derived category of k*-modules, where* Σ *is the suspension functor and* m_P *is the simplicial maximal ideal of* P.

(ii) *It is possible to choose* P *to be minimal in the sense that all the differentials of the normalized subcomplex of* m_P/m_P^2 *are zero. The completion* $\hat{P}$ *of such a minimal* P *is unique up to noncanonical isomorphism.*

Such a P as in (i) will be called a simplicial regular local ring resolution of A. It follows from 10.2 that the number of nondegenerate parameters in dimension $q-1$ of P is at least $\dim_k D_q(k/A)$ with equality if P is minimal.

If A is an arbitrary local noetherian ring, then its completion $\hat{A}$ is flat as an A-module, so by the flat base change theorem we have that $\mathbf{L}_{k/A} \simeq \mathbf{L}_{k/\hat{A}}$. Choosing a simplicial regular local ring resolution P of $\hat{A}$ and filtering P by the ideals m_P^n, we obtain a spectral sequence:

PROPOSITION 10.3. *If A is a local noetherian ring with maximal ideal m_A and residue field k, there is a spectral sequence of anticommutative k-algebras with*

$$E^2_{pq} = H_{p+q}\{S^k_q(\Sigma^{-1}\mathbf{L}_{k/A})\}$$

and

$$\begin{aligned} E^\infty_{pq} &= 0 && \text{if } p+q \neq 0 \\ &= m_A^q/m_A^{q+1} && \text{if } p+q = 0. \end{aligned}$$

This spectral sequence is not a first quadrant spectral sequence. To remedy this defect, define the Koszul homology $K_*(A)$ of A to be the homology of the Koszul complex $K.(\mathbf{x}, A)$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is a minimal system of generators for m_A. This Koszul homology depends only on A and is isomorphic to $\operatorname{Tor}^R_*(k, A)$ if A is the quotient of a regular local ring R such that $m_R/m_R^2 \simeq m_A/m_A^2$. If P is a simplicial regular local ring resolution of $\hat{A}$, then filtering $P \otimes_{P_0} k$ by powers of its augmentation ideal yields a spectral sequence relating the cotangent complex and Koszul homology.

PROPOSITION 10.4. *There is a first quadrant spectral sequence of k-algebras*

$$E^2_{pq} = H_{p+q}S^k_q\{\Sigma^{-1}T_{\geq 2}\mathbf{L}_{k/A}\} \Rightarrow K_{p+q}(A)$$

where $T_{\geq 2}\mathbf{L}_{k/A}$ is the "Postnikov" subcomplex of $\mathbf{L}_{k/A}$ with the same homology groups in dimensions ≥ 2 and zero homology groups in dimensions ≤ 1. This spectral sequence has the following properties:

Low dimensional isomorphism:

(10.5) $$K_1(A) \simeq D_2(k/A)$$

Edge homomorphisms:

(10.6) $$K_n(A) \to D_{n+1}(k/A)$$

(10.7) $$\Lambda^k_n D_2(k/A) \to K_n(A)$$

where 10.6 *annihilates the decomposable elements of K_* (A) and where* 10.7 *is the unique algebra homomorphism extending the isomorphism* 10.5.

Five term exact sequence:

$$K_3(A) \to D_4(k/A) \to \Lambda^2 K_1(A) \to K_2(A) \to D_3(k/A) \to 0.$$

COROLLARY 10.8. *A is a "local-complete-intersection" local ring* **[24]** *if and only if the map $\Lambda^2 K_1(A) \to K_2(A)$ induced by the algebra structure of $K_*(A)$ is surjective.*

11. **Euler characteristics for graded rings.** Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded ring and let B be a graded A-algebra. Then the cotangent complex has a natural structure as a simplicial graded B-module, for it may be calculated using simplicial cofibrant resolutions in the category of graded A-algebras. Consequently

$$D_q(B/A) = \bigoplus_{n=0}^{\infty} D_q(B/A)_n$$

is a graded B-module.

THEOREM 11.1. *Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded ring such that A_0 is a field k and such that each A_n is a finite-dimensional vector space over k. Then $D_q(k/A)_n$ is finite-dimensional for each q and n and*

$$D_q(k/A)_n = 0 \quad \text{for } n < q, \tag{11.2}$$

so that the integers

$$\chi_n = \sum_q (-1)^q \dim_k D_q(k/A)_n$$

are defined. Furthermore there is an identity of formal power series

$$\sum_{n \geq 0} (\dim_k A_n) t^n = \prod_{n \geq 1} (1 - t^n)^{\chi_n}. \tag{11.3}$$

This is proved by analyzing the fundamental spectral sequence 6.3 with respect to the grading induced by that of A. The key point is the following connectivity assertion for the symmetric algebra functor.

LEMMA 11.4. *If M is a simplicial k-module such that $H_q(M) = 0$ for $q > n$, then $H_q(S_m^k M) = 0$ for $q > mn$.*

When A is a graded local noetherian ring with residue field k and $D_q(k/A) = 0$ for q sufficiently large, 11.3 becomes an identity of rational functions of t. Equating orders of the pole at $t = 1$ we have the

COROLLARY 11.5. *Let A be a graded local noetherian ring with residue field k. Suppose that $D_q(k/A) = 0$ for q sufficiently large or equivalently that $\mathbf{L}_{k/A}$ is of finite projective dimension as a complex of k-modules. Then*

$$-\dim A = \sum_q (-1)^q \dim_k D_q(k/A)$$

where $\dim A$ *is the dimension of A* **[22]**.

REMARK 11.6. We do not know if the corollary remains valid without the hypothesis that A be graded. This would follow from the following special case of the conjecture 5.6:

Conjecture 11.7. If A is a local noetherian ring with residue field k and if $D_q(k/A) = 0$ for q sufficiently large, then A is a local-complete-intersection.

EXAMPLE 11.8. Let A be the graded Artin ring $A = k \oplus V \oplus k \oplus 0 \ldots$, where the multiplication is given by a nondegenerate quadratic form on the k-vector space V. Then A is a Gorenstein ring, however the Poincaré series of A,

$1 + (\dim V)t + t^2$, for $\dim V \geq 3$ has roots which are not roots of unity. Thus by 11.3 there are infinitely many nonzero χ_n and so $D_q(k/A) \neq 0$ for infinitely many q.

Since previous work in local algebra has not produced a class of local noetherian rings intermediate between the local-complete-intersection and Gorenstein rings, we consider the above example evidence for the conjecture 11.7.

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