Logarithmic Spaces
(According to K. Kato)

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0. Introduction.

The purpose of this talk is to give a survey of some of Kato's work on the theory of logarithmic structures. The notion itself is due to Fontaine and myself, but essentially the whole theory has been carried out by Kato alone. The basic ideas stemmed out of a desire to give a unified treatment of the various constructions of de Rham complexes with logarithmic poles. Such complexes have been used extensively in complex algebraic geometry since Deligne's fundamental work on differential equations and Hodge theory [9], [10]. They were also considered, to some extent, in positive characteristic, especially in the beautiful work of Katz on the Gauss-Manin connexion [33], [34], but it is only fairly recently that variants in mixed characteristic have been studied, originally by Hyodo [23] and Kato [28], with rather different motivations. In the spring of 1988, Fontaine was running a seminar at the IHES [17] on the theory of p-adic periods. Hyodo's article [23] was discussed, and the efforts to better understand his work led to the construction of a monodromy operator on de Rham cohomology in the case of semi-stable reduction in mixed characteristic, as predicted by conjectures of Jannsen (see [17]). That eventually gave birth to the notions of logarithmic structures.

Let me briefly recall what the main motivating question was. Suppose S is the spectrum of a complete discrete valuation ring A, with closed (resp. generic) point s (resp. η), and X/S is a scheme with semi-stable reduction, which means that, locally for the étale topology, X is isomorphic to the closed subscheme of $\mathbf{A}^n_S$ defined by the equation $x_1 \cdots x_r = t$, where $x_1, \ldots, x_n$ are coordinates on $\mathbf{A}^n$ and t is a uniformizing parameter of A. Then X is regular, X_η is smooth, and $Y = X_s$ is a divisor with normal crossings on X. In this situation, one can consider, with Hyodo, the relative de Rham complex of X over S with logarithmic poles along Y,

$$(0.1) \qquad \omega^\cdot_{X/S} = \Omega^\cdot_{X/S}(\log Y)$$

([23], see also [24], [25]). Its restriction to the generic fiber is the usual de Rham complex $\Omega^\cdot_{X_\eta/\eta}$, and it induces on Y a complex

$$(0.2) \qquad \omega^\cdot_Y = \mathcal{O}_Y \otimes \Omega^\cdot_{X/S}(\log Y).$$

One has $\omega^i_Y = \Lambda^i \omega^1_Y$, and when X is defined as above by $x_1 \cdots x_r = t$, ω^1_Y is generated as an $\mathcal{O}_Y$-module by the images of the $d\log x_i$ ($1 \leq i \leq r$) and dx_i ($r+1 \leq i \leq n$), subject to the single relation $\Sigma_{1 \leq i \leq r} d\log x_i = 0$. The analogue of $\omega^\cdot_Y$ over $\mathbf{C}$ (S replaced by a disc) is the complex studied by Steenbrink in [41], which "calculates" $R\Psi(\mathbf{C})$. If X/S is of relative dimension 1, ω^1_Y is simply the dualizing sheaf of Y (which probably explains the notation ω, chosen by Hyodo), and of course, in this case, $\omega^\cdot_Y$ depends only on Y. In

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general, however, $\omega^{\cdot}_Y$ depends not only on Y but also "a little bit" on X, so it is natural to ask :

(0.3) *which extra structure on* Y *is needed to define* $\omega^{\cdot}_Y$?

Assume for simplicity that Y has simple normal crossings, i.e. Y is a sum of smooth divisors Y_i meeting transversally. Let L_i be the invertible sheaf $\mathcal{O}_Y \otimes \mathcal{O}_X(-Y_i)$, and $s_i : L_i \to \mathcal{O}_Y$ the map deduced by extension of scalars from the inclusion $\mathcal{O}_X(-Y_i) \hookrightarrow \mathcal{O}_X$. Then it is easily seen that the data consisting of the pairs (L_i, s_i), together with the isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \otimes L_i$ (coming from $Y = \Sigma Y_i$), suffice to recover ω^1_Y. Indeed, ω^1_Y can be defined as the $\mathcal{O}_Y$-module generated locally by elements $\{dx, d\log e_i\}$, for x a local section of $\mathcal{O}_Y$ and e_i a local generator of L_i, subject to the usual relations among the dx, plus

$$(0.4) \qquad \begin{gathered} s_i(e_i)d\log e_i = ds_i(e_i) \\ \Sigma d\log e_i = 0 \text{ when } 1 \in \mathcal{O}_Y \text{ is written } \otimes e_i \ . \end{gathered}$$

This construction and a subsequent axiomatic development was proposed by Deligne in [11], and independantly by Faltings in [14]. In order to deal with the general case (Y no longer assumed to have simple normal crossings), it is more convenient to consider, instead of the pairs (L_i, s_i), the sheaf of monoids M on Y (multiplicatively) generated by $\mathcal{O}^*_Y$ and the e_i together with the map $M \to \mathcal{O}_Y$ given by $\alpha(\otimes e_i^{n_i}) = \Pi s_i(e_i)^{n_i}$ (replace $d\log e_i$ by $d\log e$ for $e \in M$, and the relations (0.4) by $d\log(ef) = d\log e + d\log f$, $\alpha(e)d\log e = d\alpha(e)$, $d\log u = u^{-1}du$ if u is a unit). This is the origin of the notion of logarithmic structure, proposed by Fontaine and the speaker, which gave rise to the whole theory beautifully developed by Kato (and Hyodo) in [29], [25], [30], [31], [32].

We recall in § 1 the basic facts of the theory, which can be thought of as some kind of enlargement of classical algebraic geometry. One remarkable phenomenon discovered by Kato is that some large classes of singular schemes or morphisms, once equipped with suitable logarithmic structures, behave as being smooth.

It is probably too early to estimate the real importance of these new concepts, but there are already a few striking applications. The first one, which we discuss in § 2, is the construction by Hyodo and Kato of the monodromy operator alluded to above, and the proof by Kato (under some restriction on the dimension) of the C_{st} conjecture of Fontaine and Jannsen. As we have just said, the theory of logarithmic structures introduces a new range of smoothness, and therefore sheds a new light on classical problems of degenerations or compactifications. For example, generalized elliptic curves become proper and smooth group objects in what Kato calls the "logarithmic world". Hence one can consider their points or order n, their corresponding "logarithmic" Barsotti-Tate groups, their "logarithmic" Dieudonné modules, etc. In § 3, we briefly outline Kato's work on these questions. See also Messing's expcsé in this volume for more details on Kato's work on Dieudonné theory.

I am grateful to W. Bauer and A. Ogus for their comments on a first version of this text.

1. Logarithmic structures.

1.0. We shall have to deal with monoids (= semi-groups), which will be assumed to be associative, commutative, an have a unit; all morphisms are assumed to send the unit to the unit. The group envelope of a monoid P is denoted P^{gp}. A monoid P is called *integral* if $ab = ac$ in P implies $b = c$, or equivalently if the canonical map $P \to P^{gp}$ is injective.

1.1. Let $(X, \mathcal{O}_X)$ be a ringed space (or topos). A *pre-logarithmic* (or *pre-log*) *structure* on X is a pair (M, α) consisting of a sheaf of monoids M on X and a homorphism $\alpha : M \to \mathcal{O}_X$ (where $\mathcal{O}_X$ is considered as a sheaf of monoids via the multiplication). A pre-log structure (M, α) is said to be a *logarithmic* (or *log*) *structure* if α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*$.

A ringed space endowed with a pre-log (resp. log) structure is called a *pre-log space* (resp. *log space*), and is usually denoted by (X, M). A map of pre-log spaces $f : (X, M) \to (Y, N)$ is a pair (f, u), where $f : X \to Y$ is a map of ringed spaces and $u : f^{-1}(N) \to M$ is a map of sheaves of monoids such that the square

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & \mathcal{O}_X \\ {\scriptstyle u}\uparrow & & \uparrow \\ f^{-1}(N) & \xrightarrow{\alpha} & f^{-1}(\mathcal{O}_Y) \end{array}$$

commutes. A map of log-spaces is a map of the underlying pre-log spaces.

If X is any ringed space, the inclusion $\mathcal{O}_X^* \to \mathcal{O}_X$ is a log structure on X, called the *trivial log structure*. If $\beta : P \to \mathcal{O}_X$ is a pre-log structure on X, there is defined a log structure $\alpha : M \to \mathcal{O}_X$ together with a map from (P, β) to (M, α) (i.e. a map $f : P \to M$ such that $\alpha f = \beta$), which is universal for maps from (P, β) to log structures. The log structure (M, α), which is unique up to unique isomorphism, is called the *log structure associated to* (P, β), and denoted by $(P \to \mathcal{O}_X)^a$ (or P^a). M can be defined as the push-out of

$$\begin{array}{ccc} \beta^{-1}(\mathcal{O}_X^*) & \longrightarrow & P \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on X, with the map $M \to \mathcal{O}_X$ corresponding to the pair of β and the inclusion of $\mathcal{O}_X^*$ into $\mathcal{O}_X$.

If $f : X \to Y$ is a map of ringed spaces and $\beta : N \to \mathcal{O}_X$ is a log structure on Y, the *inverse image log structure* on X is defined as the log structure associated to $f^{-1}(N) \xrightarrow{f^{-1}(\beta)} f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, and denoted by $f^*N \to \mathcal{O}_X$.

Although it would be interesting to consider log structures on analytic spaces (either classical or rigid), so far the theory has been developed only on schemes. When dealing with schemes, we shall work with the étale topology, unless otherwise stated. The log structures encountered in practise enjoy certain constructibility properties. The following basic notion, introduced by Kato, seems to cover most of the cases.

1.2. A log structure (M, α) on a scheme X is said to be *fine* if locally (for the étale topology) (M, α) is associated to a pre-log structure of the form (P_X, β), where P_X is a constant sheaf of monoids of value P, with P finitely generated and integral. A scheme endowed with a fine log structure is called a *fine log scheme*. If (X, M) is a fine log scheme, a *chart of* M is a homomorphism $P_X \to M$, with P finitely generated and integral, inducing an isomorphism $(P_X)^a \xrightarrow{\sim} M$. If $f : (X, M) \to (Y, N)$ is a map of fine log schemes, a *chart of* f is a triple $c : (P_X \to M, Q_Y \to N, Q \to P)$ where $P_X \to M$, $Q_Y \to N$ are charts of X and Y resp., and $Q \to P$ is a homomorphism making the diagram

$$\begin{array}{ccc} M & \leftarrow & P_X \\ \uparrow & & \uparrow \\ f^{-1}(N) & \leftarrow & Q_X \end{array}$$

commutative.

Here are some basic examples.

1.3. (a) **Monoid algebras.** Let R be a ring, P a monoid, and T the spectrum of the monoid algebra $R[P]$. The log structure on T associated to the pre-log structure given by the inclusion $P \to R[P]$ is called the canonical log structure on T. If (X, M) is a fine log scheme and $P_X \to M$ is a chart of X, then this chart defines a map $X \to \operatorname{Spec} \mathbb{Z}[P]$, and the log structure on X is the inverse image of the canonical log structure of $\operatorname{Spec} \mathbb{Z}[P]$.

(b) **Normal crossings divisors.** Let X be a regular, locally noetherian scheme, Y a reduced divisor with normal crossings on X (i.e. étale locally of the form $V(t_1 \cdots t_r)$ where $t_1, \ldots, t_r$ are part of a regular system of parameters at a point), and $j : U = X - Y \to X$ the inclusion. Then the inclusion

$$M = \mathcal{O}_X \cap j_* \mathcal{O}_U^* \to \mathcal{O}_X$$

is a fine log structure on X called the canonical log structure on X defined by Y. If as above $t_1 \cdots t_r$ is a local equation of Y near a geometric point x, then

$$\mathbb{N}^r \to M \ , \ (n_i) \mapsto \Pi t_i^{n_i}$$

is a chart of M at x.

(c) **Log points.** Let s be the spectrum of a field k. If P is a finitely generated and integral monoid whose subset of invertible elements $P^{\times} = \{1\}$ (multiplicative notations), then the map

$$P \to k \ , \ a \mapsto 0 \text{ (resp.1) if } a \neq 1 \text{ (resp. } a = 1)$$

is a chart of a (fine) log structure M on s such that $M/\mathcal{O}_s^* = P$ (conversely, it is easily seen that, if k is algebraically closed, any fine log structure on s is of this form). Such log schemes are called log points. Especially important is the log point corresponding to $P = \mathbf{N}$, which is called the ***standard log point***. It is the analogue of the punctured disc, so it should be thought of as a "punctured point"! Indeed, if S is the spectrum of a discrete valuation ring with residue field k, then the canonical log structure on S defined by the closed point s as in (b) induces on s the log structure of the standard log point, and notions such as semi-stable reduction, tame ramification can be rephrased purely in terms of the log point (see [31, 4.3] for a discussion of tame fundamental groups in this context).

(d) **Semi-stable reduction.** Let S be the spectrum of a discrete valuation ring and X/S a scheme with semi-stable reduction (cf. the introduction), with special fiber $X_s = Y$. Then the divisor with normal crossings Y in X (resp. s in S) defines a log structure M_X (resp. M_S) on X (resp. S) as in (b), and a map of log schemes $\underline{X} \to \underline{S}$(1) Near a point x of Y this map has locally a chart of the form ($\mathbf{N}^r \to M_X$; $\mathbf{N} \to M_S; \mathbf{N} \to \mathbf{N}^r, 1 \mapsto (1, \ldots, 1)$). Such a chart can be chosen such that the corresponding map $f : X \to S \times_{S[t]} S[t_1, \ldots, t_r]$ $(t \mapsto t_1 \cdots t_r)$ be smooth and $f^* t_i$ vanish at x; in this case, r is the "number of branches" of Y at x.

1.4. (a) If $\underline{X} = (X, M_X)$ is a fine log scheme whose underlying scheme X is locally noetherian, $M_X/\mathcal{O}_X^*$ is a constructible sheaf of monoids on X, i.e. X is locally a disjoint union of locally closed subschemes over which $M_X/\mathcal{O}_X^*$ is locally constant, of fibers a finitely generated and integral monoid. For example, in the case of 1.3 (b), if Y is assumed to have simple normal crossings, i.e. is the sum of regular divisors Y_i crossing transversally, then $M_X/\mathcal{O}_X^* = \oplus \mathbf{N}_{Y_i}$, and the knowledge of the extension

$$0 \to \mathcal{O}_X^* \to M_X \to M_X/\mathcal{O}_X^* \to 0$$

is equivalent to that of the cohomology classes $cl(Y_i) \in H^1_{Y_i}(X, \mathcal{O}_X^*)$ of the components Y_i.

(b) The category of log schemes admits finite inverse limits. So does the full subcategory of fine log schemes, but the limit in this subcategory does not commute in general with the forgetful functor, since an amalgamated sum $P_1 \oplus_Q P_1$ of finitely generated and integral monoids need not be integral.

(1) From now on we shall usually denote a log scheme by an underlined letter $\underline{X}$, the non underlined letter X then meaning the underlying scheme.

1.5. Differentials. Let $\underline{X} = (X, M_X) \xrightarrow{f} (\underline{Y} = (Y, M_Y)$ be a morphism of log schemes. Consider triples $(E, D, D\log)$ where E is an $\mathcal{O}_X$-module, $D : \mathcal{O}_X \to E$ is a Y-derivation, and $D\log : M_X \to E$ a map satisfying the conditions :

(i) $D\log(ab) = D\log a + D\log b$

(ii) $\alpha(a)D\log a = D\alpha(a)$ (where $\alpha : M_X \to \mathcal{O}_X$ is the structural map)

(iii) $D\log\phi(c) = 0$ (where $\phi : f^{-1}M_Y \to M_X$ is the structural map), for all local sections a, b of M_X and c of M_Y. The functor associating to an $\mathcal{O}_X$-module E the set of pairs $(D, D\log)$ satisfying (i) - (iii) is representable by an $\mathcal{O}_X$-module of "Kähler differentials"

$$\Omega^1_{\underline{X}/\underline{Y}}, \tag{1.5.1}$$

sometimes also denoted $\omega^1_{X/Y}$ (if no confusion on the log structures can arise), together with a universal pair

$$d : \mathcal{O}_X \to \Omega^1_{\underline{X}/\underline{Y}}\ ,\ d\log : M_X \to \Omega^1_{\underline{X}/\underline{Y}}\ .$$

This module is realized in an obvious way as a quotient of $\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} \mathrm{Coker}(f^{-1}M_Y^{gp} \to M_X^{gp}))$. When $\underline{X}$ has the inverse image log structure, one simply has $\Omega^1_{\underline{X}/\underline{Y}} = \Omega^1_{X/Y}$. The formation of Ω^1 commutes with base changes and a composition $\underline{X} \xrightarrow{f} \underline{Y} \to \underline{Z}$ gives rise to an exact sequence

$$f^*\Omega^1_{\underline{Y}/\underline{Z}} \to \Omega^1_{\underline{X}/\underline{Z}} \to \Omega^1_{\underline{X}/\underline{Y}} \to 0\ . \tag{1.5.2}$$

There is a unique way to extend d into a derivation of the exterior algebra

$$\Omega^{\cdot}_{\underline{X}/\underline{Y}} = \Lambda\Omega^1_{\underline{X}/\underline{Y}} \tag{1.5.3}$$

satisfying $d^2 = 0$ and $d(d\log) = 0$. The resulting complex is called the *de Rham complex* of $\underline{X}/\underline{Y}$. For example, in the case of semi-stable reduction 1.3 (d) $\Omega^{\cdot}_{\underline{X}/\underline{S}}$ is the complex (0.1), and if $\underline{Y}/\underline{s}$ is the pull-back of $\underline{X}/\underline{S}$, i.e. Y/s is equipped with the log structures induced by those of $\underline{X}/\underline{S}$, then $\Omega^{\cdot}_{\underline{Y}/\underline{s}}$ is the complex (0.2).

1.6. Log smoothness. Classically, smoothness is the conjunction of "locally finite presentation" and "formal smoothness". One can define log smoothness along the same lines.

In order to define log formal smoothness, one needs an analogue of the notion of thickening in the log context. A map $i : \underline{S} \to \underline{T}$ of fine log schemes is called a *thickening* if the underlying map of schemes is a closed immersion defined by a nilideal I and the map $i^*M_T \to M_S$ is an isomorphism; it is called a *thickening of order n* if $I^{n+1} = 0$. A map $f : \underline{X} \to \underline{Y}$ of fine log schemes is called *smooth* (resp. *étale*) if the underlying map of schemes is locally of finite presentation and if for every commutative square of fine log schemes

$$\begin{array}{ccc} \underline{S} & \longrightarrow & \underline{X} \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ \underline{T} & \longrightarrow & \underline{Y} \end{array}$$

where i is a thickening of order 1, there exists étale locally on T (1) a dotted arrow (resp. a unique dotted arrow) making the whole diagram commute.

A typical example of a log smooth (resp. étale) map is the morphism of log schemes $\operatorname{Spec} R[P] \to \operatorname{Spec} R[Q]$ associated to a ring R and a map $Q \to P$ of finitely generated and integral monoids (1.3 (a)) such that the kernel and the torsion part of the cokernel (resp. the cokernel) of $Q^{gp} \to P^{gp}$ (cf. 1.0) are annihilated by integers invertible on $\operatorname{Spec} R$. In general, a log smooth (resp. étale) map is locally obtained by base change and smooth (or étale) localization from such a map. Indeed, Kato gives the following convenient criterion :

1.7. *A map $f : \underline{X} \to \underline{Y}$ of fine log schemes is (log) smooth (resp. étale) if and only if étale locally on X and Y (in the usual sense) there exists a chart $c = (P_X \to M_X, Q_Y \to M_Y, Q \to P)$ of f (1.2) such that*

(i) *the kernel and the torsion part of the cokernel (resp. and the cokernel) of $Q^{gp} \to P^{gp}$ are annihilated by an integer invertible on X*

(ii) *the morphism*

$$X \to Y \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$$

induced by c (cf. 1.3 (a)) is smooth (resp. étale) in the usual sense.

Examples 1.8. (a) If $\underline{X}$ is log smooth over an algebraically closed field k endowed with the trivial log structure, then étale locally X is étale over $\operatorname{Spec} k[P]$, where P is a finitely generated and integral monoid, such that $P^{gp} = \mathbf{Z}^r \oplus \oplus_i \mathbf{Z}/n_i\mathbf{Z}$ with n_i prime to char k; thus $\underline{X}$ gives to a (generalized) *toroidal embedding* in the sense of [36] (in loc. cit. P^{gp} is assumed to be torsionfree).

(b) The map $\underline{X} \to \underline{S}$ of 1.3 (d) is (log) smooth.

1.9. Log smooth (resp. étale) maps enjoy most of the properties of classical smooth (resp. étale) maps. In particular, they are stable under composition and base change (in the category of fine log schemes). If $\underline{X} \to \underline{Y}$ is log smooth, then $\Omega^1_{\underline{X}/\underline{Y}}$ is locally free of finite type (if c is a chart as in 1.7, then $\Omega^1_{\underline{X}/\underline{Y}} = \mathcal{O}_X \otimes \operatorname{Coker}(Q^{gp} \to P^{gp})$). If $\underline{X} \xrightarrow{f} \underline{Y} \to \underline{Z}$ is a composition of fine log schemes and f is smooth, then one can add a zero on the left to the sequence (1.5.2).

(1) a posteriori, this is equivalent to "Zariski locally"

Note that the map of schemes underlying a log smooth (or even étale) map is not necessarily flat : for example, if S is any base scheme, the piece of the blow-up

$$f : \underline{X} = \mathbb{A}^2_S = S[x,y] \to \underline{Y} = \mathbb{A}^2_S = S[x,y] \quad (x,y) \mapsto (x,xy) ,$$

where $\mathbb{A}^2_S$ is given the canonical log structure 1.3 (a), is log étale, but of course not flat. In this example log differentials behave much better than usual differentials : if D is the divisor $xy = 0$, then $\Omega^1_{\underline{X}/S} = \Omega^1_{X/S}(\log D)$, $\Omega^1_{\underline{Y}/S} = \Omega^1_{Y/S}(\log D)$, and the natural map $f^*\Omega^1_{Y/S}(\log D) \to \Omega^1_{X/S}(\log D)$ is an isomorphism.

Beautiful and simple as they may appear, log smooth maps are far from being well understood : while toroidal embeddings have been extensively studied, the structure of general log smooth maps $\underline{X} \to \underline{Y}$ has not yet been investigated, even when $\underline{Y}$ is a log point.

2. **Log crystalline cohomology.**

2.1. It is easy to transpose the basic definitions and results of crystalline cohomology [2] into the context of log schemes. Let $\underline{X} \to \underline{S}$ be a morphism of fine log schemes, and (I, γ) a PD-ideal on S (i.e. a quasi-coherent ideal of $\mathcal{O}_S$ equipped with a divided power structure γ) such that γ extends to X ; assume some prime number p is nilpotent on S. The objects of the log crystalline site of $\underline{X}$ over $\underline{S}$,

$$(\underline{X}/\underline{S})_{\mathrm{crys}} ,$$

are PD-thickenings $\underline{U} \to \underline{T}$ over $\underline{S}$, i.e. $\underline{S}$-thickenings (1.3) of fine log schemes such that the ideal J of $U \to T$ is equipped with a PD-structure compatible with γ, where U is étale over X and endowed with the induced log structure; maps are defined in the obvious way and coverings are taken for the étale topology of the underlying schemes. The corresponding topos depends functorially on $\underline{X}/\underline{S}$. One defines as usual the structural sheaf of rings $\mathcal{O} = \mathcal{O}_{\underline{X}/\underline{S}}$, $(\underline{U} \to \underline{T}) \mapsto \mathcal{O}_T$, and the notion of crystal in $\mathcal{O}$-modules on $\underline{X}/\underline{S}$ (a sheaf of $\mathcal{O}$-modules E on $(\underline{X}/\underline{S})_{\mathrm{crys}}$ given by $(\underline{U} \to \underline{T}) \mapsto E_T$ such that for any $g : (\underline{U}' \to \underline{T}') \to (\underline{U} \to \underline{T})$ in $(\underline{X}/\underline{S})_{\mathrm{crys}}$, $g^*E_T \to E_{T'}$, is an isomorphism). There is also a notion of divided power envelope $D_{\underline{X}}(\underline{Z})$ of a closed immersion $\underline{X} \xrightarrow{i} \underline{Z}$ (i.e. a map of fine log schemes such that the underlying map of schemes is a closed immersion and $i^*M_Z \to M_X$ is surjective). If $i : \underline{X} \to \underline{Z}$ is a closed immersion of $\underline{X}$ into a (log) smooth $\underline{S}$-scheme, and if $\underline{D}$ is the PD-envelope of i, then a crystal E on $\underline{X}/\underline{S}$ can be described by an $\mathcal{O}_D$-module E_D on $D_{\text{ét}}$, endowed with an integrable connection relative to $\Omega^{\cdot}_{\underline{Z}/\underline{S}}$

$$\nabla : E_D \to E_D \otimes_{\mathcal{O}_Z} \Omega^1_{\underline{Z}/\underline{S}} ,$$

satisfying a certain nilpotence condition (see [29, 6.2] for details). Finally, the de Rham complex of $\underline{Z}/\underline{S}$ with coefficients in E_D calculates the crystalline cohomology of E, i.e., if

$$u_{\underline{X}/\underline{S}} : (\underline{X}/\underline{S})_{\mathrm{crys}} \to \underline{X}_{\text{ét}}$$

is the map of topoi defined as in the classical case by $u_{\underline{X}/\underline{S}_\bullet}(E)(U) = \Gamma(\underline{U}/\underline{S}_{\text{crys}}, E)$, then there is a canonical isomorphism in $D(X, f^{-1}\mathcal{O}_S)$ (where $f : X \to S$ is the structural map)

$$Ru_{\underline{X}/\underline{S}_\bullet}(E) \simeq E_D \otimes \Omega^\cdot_{\underline{X}/\underline{S}} .$$

The proof of these facts is not as straightforward as it might look : it indeed involves a subtle generalization of the divided power Poincaré lemma to the log context, see [29, 6.4].

2.2. As we have said in the introduction, the main application of this formalim is the construction of a monodromy operator on the de Rham cohomology of schemes with semi-stable reduction in mixed characteristic. This is done in two steps.

The first one is the construction of a nice cohomology theory for proper and (sufficiently) smooth log schemes over the log point. Here is a sketch.

Let k be a perfect field of characteristic $p > 0$ and let $\underline{S}$ be the standard log point (cf. 1.3 (c)) defined by $S = \operatorname{Spec} k$ endowed with the log structure associated to $\mathbb{N} \to k$, $1 \mapsto 0$. If n is an integer ≥ 1, denote by $W_n(\underline{S})$ the log scheme defined by $\operatorname{Spec} W_n(k)$ endowed with the log structure associated to $\mathbb{N} \to W_n(k)$, $1 \mapsto 0$. For a smooth log scheme $\underline{Y}/\underline{S}$ (1.5), one defines

$$H^i(\underline{Y}/W_n(\underline{S})) := H^i(\underline{Y}/W_n(\underline{S}))_{\text{crys}}, \mathcal{O}) \tag{2.2.1}$$

(where $W_n(k)$ is equipped with the standard PD-structure on (p)). This is a $W_n(k)$-module. It is endowed with a σ-linear operator ("*Frobenius*")

$$\phi : H^i(\underline{Y}/W_n(\underline{S})) \to H^i(\underline{Y}/W_n(\underline{S}))$$

(where σ is the Frobenius automorphism of W_n) and a linear operator ("*monodromy*")

$$N : H^i(\underline{Y}/W_n(\underline{S})) \to H^i(\underline{Y}/W_n(\underline{S})),$$

related by

$$N\phi = p\phi N \ , \tag{2.2.2}$$

and which are defined as follows. There is an absolute Frobenius map

$$F_{abs} : \underline{Y}/W_n(\underline{S}) \to \underline{Y}/W_n(\underline{S}) \ ,$$

given on the underlying schemes by the absolute Frobenius and on the monoids by raising elements to the p^{th}-power (in the multiplicative notation). The operator ϕ is obtained by functoriality from F_{abs}. This definition does not use the smoothness of $\underline{Y}/\underline{S}$. The definition of N does use it. Let $f : \underline{Y} \to \underline{S}$ be the structural map. Denote by $W_n(S)$ the scheme $\operatorname{Spec} W_n(k)$ endowed with the *trivial* log structure. Then, imitating the arguments in Berthelot's thesis [2, V], one shows that $Rf_*\mathcal{O}_{\underline{Y}/W_n(S)}$ is a "crystal in objects of the derived category" on $\underline{S}/W_n(S)$, from which $H^i(\underline{Y}/W_n(\underline{S}))$ is obtained by evaluating at the thickening $W_n(\underline{S})$ of $\underline{S}$ and applying H^i. Using the embedding of $\underline{S}$ in the affine line $A = \operatorname{Spec} W_n(k)[t]$ endowed with the log structure given by the divisor $(t = 0)$, one describes this crystal structure by a complex on the divided power envelope of A, equipped with what plays the role of a connection with log poles at $t = 0$. Taking the residue at 0 and applying H^i yields the desired map N. The relation (2.2.2) follows from the compatibility of the connection with Frobenius. For details, see [25].

For varying n, the W_n-modules (2.2.1) form an inverse system, and the operators ϕ and N are compatible with the transition maps. They induce operators again denoted ϕ and N on

$$H^i(\underline{Y}/W_n(\underline{S})) := \varprojlim H^i(\underline{Y}/W_n(\underline{S})) , \tag{2.2.3}$$

related by (2.2.2). If $\underline{Y}/\underline{S}$ satisfies a certain additional condition, which Kato calls "being of Cartier type" [1] , and which is verified for example if $\underline{Y}/\underline{S}$ comes by reduction over the closed point of a scheme with semi-stable reduction over a discrete valuation ring (cf. 1.3 (d)), then ϕ on (2.2.3) is an isogeny, i.e. $\phi \otimes \mathbf{Q}$ is bijective. If moreover Y is proper over S, then $H^i(\underline{Y}/W(\underline{S}))$ is finitely generated over W, and (2.2.2) then implies that $N \otimes \mathbf{Q}$ is nilpotent.

Let us call for short a log scheme $\underline{Y}/\underline{S}$ *good* if it is smooth, of Cartier type, and Y is proper over S. Let K be the fraction field of W. On good log schemes $\underline{Y}/\underline{S}$, $H^*(\underline{Y}/W(\underline{S})) \otimes K$ is a cohomology theory with values in the category of (graded) F-isocrystals over k, endowed with a linear (nilpotent) operator N related to ϕ by (2.2.2). It satisfies Poincaré duality and Künneth formulas, there is even a de Rham-Witt theory available [25]. However, Chern and cycle classes, trace formulas are missing so far, as well as comparison with étale cohomology, in the line of Katz-Messing [35].

The second step is a generalization of the Berthelot-Ogus comparison theorem between crystalline and de Rham cohomology [3].

Let A be a complete discrete valuation ring with perfect residue field k of characteristic $p > 0$ and field of fractions K of characteristic 0. Let $\underline{T}$ be the log scheme defined by $\operatorname{Spec} A$, endowed with the log structure given by the inclusion of the closed point $S = \operatorname{Spec} k$, as in 1.3 (b). Let $\underline{S}$ be the log scheme defined by S with the log structure induced by that of $\underline{T}$, or equivalently associated to $\mathbf{N} \to k$, $1 \mapsto 0$. Let $\underline{X}$ be a smooth log scheme over $\underline{T}$ such that X/T is proper and the special fiber $\underline{Y} = \underline{X} \times_{\underline{T}} \underline{S}$ is of Cartier type (a typical example is the log scheme defined by a proper scheme X over T with semi-stable reduction, endowed with the log structure defined by the special fiber). Let $\underline{X}_K$ be the generic fiber, which is log smooth over $\operatorname{Spec} K$ endowed with the trivial log

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[1] This means that the relative Frobenius map $\underline{Y} \to \underline{Y}'$ (where $\underline{Y}'$ is the pull-back of $\underline{Y}$ by the absolute Frobenius of $\underline{S}$) is exact; a map of integral monoids $h : Q \to P$ is said to be exact if $Q = (h^{gp})^{-1}(P)$; a map of fine log schemes $\underline{X} \xrightarrow{f} \underline{Y}$ is said to be exact if $(f^*M_Y)_{\bar{x}} \to M_{X,\bar{x}}$ is exact for every $x \in X$. For $\underline{Y}/\underline{S}$ of Cartier type there is a good theory of Cartier isomorphism [29, 4.12].

structure. Fix a uniformizing parameter t of T. In this situation, Hyodo and Kato [25] construct a canonical isomorphism

$$(2.2.4) \qquad H^i(\underline{Y}/W(\underline{S})) \otimes_W K \xrightarrow{\sim} H^i_{DR}(\underline{X}_K/K) \quad (=_{dfn} H^i(X_K, \Omega^{\cdot}_{\underline{X}_K/K}).$$

This isomorphism depends on t. Call it ρ_t. If u is a unit, it is shown in loc. cit. that

$$\rho_t = \rho_{ut} \exp(\log(u)N),$$

where log is the usual p-adic logarithm ($=0$ on k^*). When the log structure of $\underline{X}$ is induced by that of $\underline{T}$, i.e. X is smooth (in the usual sense) over T, then (2.2.4) should coincide with the isomorphism of Berthelot-Ogus (some compatibilities have to be checked). The construction of (2.2.4) is very involved. A simpler one (although using basically the same ideas) has recently been given by Ogus [40], at least in the case where A is unramified over W. Ogus's construction works more generally for any finitely generated F-crystal on $\underline{S}/W$ (or even any F-crystal in perfect complexes on $\underline{S}/W$). Assuming for simplicity that the crystals $R^*f_*\mathcal{O}_{\underline{Y}/W} = \varprojlim R^*f_*\mathcal{O}_{\underline{Y}/W_n}$ considered above are locally free, then, when K is the fraction field of W, the isomorphism (2.2.4) can be viewed as an isomorphism between the "values" (tensored with K) of $R^if_*\mathcal{O}_{\underline{Y}/W}$ at two (a priori quite unrelated) log thickenings of $\underline{S}$, $W(\underline{S})$ on the one hand, and $\operatorname{Spec} W$ with the log structure given by the closed point on the other hand (here "thickening" is taken in a slightly generalized sense, i.e. "p nilpotent" is replaced by "p-adically complete"). The former log structure is associated to $\mathbf{N} \to W$, $1 \mapsto 0$, the latter to $\mathbf{N} \to W$, $1 \mapsto p$. One can interpolate these two log structures by those given by $\mathbf{N} \to W$, $1 \mapsto p^a$, a an integer ≥ 1 (or ∞). Denote by $\underline{W}(a)$ the corresponding log scheme, which is a (generalized thickening of $\underline{S}$, and if E is a crystal on $\underline{S}/W$, denote by $E(a)$ its value at $\underline{W}(a)$. What Ogus shows (in a little more general form) is that, if E is a torsion free, finitely generated F-crystal on $\underline{S}/W$, there is a unique family of isomorphisms

$$\varepsilon_a : E(\infty) \otimes K \xrightarrow{\sim} E(a) \otimes K ,$$

which is integral for $\underline{a}$ large enough, is compatible with Frobenius, and for $\underline{a}$ large enough reduces to the obvious isomorphism $\operatorname{mod} p^a$ (the key observation is that despite the fact that there is no Frobenius endomorphism of $\underline{W}(a)$ for $a < \infty$, there is a natural Frobenius map from $\underline{W}(a)$ to $\underline{W}(pa)$).

2.3. Thanks to (2.2.4), $H^i_{DR}(\underline{X}_K/K)$ comes equipped with the following structure :

(i) the K_0-structure $H^i(\underline{Y}/W(\underline{S})) \otimes K_0$, where K_0 is the fraction field of W

(ii) the σ-linear, bijective, endomorphism ϕ of $H^i(\underline{Y}/W(\underline{S}) \otimes K_0$

(iii) the K_0-linear, nilpotent, endomorphism N of $H^i(\underline{Y}/W(\underline{S})) \otimes K_0$, related to ϕ by (2.2.2)

(iv) the Hodge filtration of $H^i_{DR}(\underline{X}_K/K)$, $\operatorname{Fil}^r H^i_{DR} = H^i(X_K, \Omega^{\geq r}_{\underline{X}_K/K})$.

In short, $H^i_{DR}(\underline{X}_K/K)$ becomes an object of the category $MF_K(\phi, N)$ of Fontaine (see [17], [18], and [26]). Fontaine's theory functorially associates to any object of $MF_K(\phi, N)$ a p-adic representation of $\mathrm{Gal}(\overline{K}/K)$ (where $\overline{K}$ is an algebraic closure of K). In the case of $H^i_{DR}(\underline{X}_K/K)$, with X as above, Fontaine conjectures that the corresponding p-adic representation should be the p-adic étale cohomology $H^i(X_{\overline{K}}, \mathbf{Q}_p)$. More precisely, he conjectures that there should be a nice, functorial "period isomorphism"

$$B_{st} \otimes_W H^i(\underline{Y}/W(\underline{S})) \xrightarrow{\sim} B_{st} \otimes_{\mathbf{Q}_p} H^i(X_{\overline{K}}, \mathbf{Q}_p) \tag{2.3.1}$$

(where B_{st} is Fontaine's ring, normalized as in [26, 1.2.3] by the uniformizing parameter t), compatible with the various structures on both sides (filtrations, actions of Galois, ϕ and N). This is the so-called *C_{st}-conjecture*. This conjecture has been proven by Kato in the case X has semi-stable reduction (and is equipped with the corresponding standard log structure) and is of relative dimension $< (p-1)/2$. The restriction of dimension is inherent to the method, which is close in spirit to that of Fontaine-Messing [19]; the reason why he has to assume X with semi-stable reduction is that the calculation of p-adic vanishing cycles, which is the key ingredient in the proof, has been done so far only in this case (by Hyodo and Kurihara [23], [37]). It might be expected that Faltings' approach [16] would yield a proof in general. See also [16] where some variants of C_{st} "with coefficients" are considered, especially over modular curves.

3. Log degeneration of commutative group schemes.

3.1. The Tate curve revisited.

Let A be a complete discrete valuation ring, with maximal ideal $\mathfrak{m}$, residue field k and fraction field K. Let $T = \operatorname{Spec} A$. Fix a uniformizing parameter t of A. Recall Raynaud's construction [13, VII 1] of the Tate curve

$$E^t$$

over A with q-invariant t. This is a generalized elliptic curve whose generic fiber E^t_K is the rigid analytic quotient $\mathbf{G}_{mK}/t^{\mathbf{Z}}$. As a formal scheme, it is the limit of proper curves E^t_n over $T_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$, which are obtained as follows. Let

$$\overline{\mathbf{G}}^t_m$$

be the "compactified" Néron-Raynaud model of $(\mathbf{G}_m)_K$ over A. This is a scheme which is locally of finite type over A, and which is a union of affine open subsets U_i ($i \in \mathbf{Z}$) isomorphic to $T[x,y]/(xy-t)$, glued together along $U_i \cap U_{i+1} \simeq \mathbf{G}_{mT}$ by the process explained in (loc. cit.) (from the rigid analytic point of view, U_i corresponds to the annulus "t^i divides $\underline{a}$ divides t^{i+1}"). The discrete group $\mathbf{Z}$ (denoted rather $t^{\mathbf{Z}}$) operates on $\overline{\mathbf{G}}^t_m$, t^i sending U_0 isomorphically to U_i. The scheme $\overline{\mathbf{G}}^t_m$ has semi-stable reduction, with special fiber an infinite string of projective lines over k. Over T_n the subgroup $t^{2\mathbf{Z}}$ acts freely and therefore the quotient by $t^{\mathbf{Z}}$ exists, this is by definition E^t_n :

$$E^t_n = (\overline{\mathbf{G}}^t_m)_{T_n}/t^{\mathbf{Z}} .$$

This is a proper scheme over T_n. The formal scheme $\varinjlim E_n^t$ can be algebraized, yielding the desired Tate curve E^t. It has semi-stable reduction, with special fiber a cubic with a rational node. Because of this singularity, the T-scheme $\overline{\mathbb{G}}_m^t$ (resp. E^t) has no group structure. Instead, if $\mathbb{G}_m^t$ (resp. $(E^t)^0$) denotes the smooth locus of $\overline{\mathbb{G}}_m^t$ (resp. E^t), there is a natural action of $\mathbb{G}_m^t$ on $\overline{\mathbb{G}}_m^t$, and, by passing to the quotient by $t^{\mathbf{Z}}$, of $(E^t)^0$ on E^t ($\mathbb{G}_m^t$ (resp. $(E^t)^0$) is the Néron-Raynaud model of $(\mathbb{G}_m)_K$ (resp. Néron model of E_K^t)).

Kato shows that in a suitable category of log spaces $\overline{\mathbb{G}}_m^t$ and E^t acquire group structures. Let $\underline{T}$ be the log scheme defined by T with the standard log structure, i.e. associated to $\mathbf{N} \to A$, $1 \mapsto t$, and denote by $\underline{t}$ the image of 1 in M_T. Let $\underline{\overline{\mathbb{G}}}_m^t$ (resp. $\underline{E}^t$) be the log scheme defined by $\overline{\mathbb{G}}_m^t$ (resp. E^t) with the log structure defined by the special fiber (1.3 (d)). This log scheme is smooth over $\underline{T}$, but it is still not possible to put a group structure on it (extending the natural one on the generic fiber). To do this, Kato imitates in the log context Raynaud's interpretation of rigid analytic spaces over K as formal schemes over A "modulo blow-ups in the special fiber".

Call an abstract monoid P *valuative* if it is integral and, for any $a \in P^{gp}$, either $a \in P$ or $a^{-1} \in P$ (this is the analogue of the notion of valuative ring). A log space Z (whose underlying space is locally ringed), is said to be *valuative* if the fibers M_z of M_Z are valuative. For example, $\underline{T}$ and its reductions $\underline{T}_n \bmod t^n$ are valuative. Let $\underline{Z}$ be a valuative log space over $\underline{T}$, denote again by $\underline{t}$ the image of $\underline{t}$ in $\Gamma(Z, M_Z)$. Coming back to the definition of $\overline{\mathbb{G}}_m^t$, it is not hard to see that the set of $\underline{T}$-morphisms from $\underline{Z}$ to $\underline{\overline{\mathbb{G}}}_m^t$ is the set of $a \in \Gamma(Z, M_Z^{gp})$ such that at any point x of Z there exists rational integers $i \leq j$ such that $\underline{t}^i | a | \underline{t}^j$ in M_x^{gp}, where $a|b$ means that $ba^{-1} \in M_x$. This set is a subgroup of $\Gamma(Z, M_Z^{gp})$. Thus $\underline{\overline{\mathbb{G}}}_m^t$ is a group-valued functor on the category of valuative log spaces over $\underline{T}$, and a similar assertion holds with $\underline{T}$ replaced by $\underline{T}_n$. Unfortunately, $\underline{\overline{\mathbb{G}}}_m^t$ (or $\underline{\overline{\mathbb{G}}}_{m\underline{T}_n}^t$) is not valuative, so this does not quite give a group structure on it. But Kato shows that for any fine log scheme $\underline{X}$ over $\underline{T}$ there exists a (unique up to unique isomorphism) valuative log space $\underline{X}^{\mathrm{val}}$ together with a map $\underline{X}^{\mathrm{val}} \to \underline{X}$ such that $\underline{X}^{\mathrm{val}}(\underline{Z}) \xrightarrow{\sim} \underline{X}(\underline{Z})$ functorially in the valuative log space $\underline{Z}$ over $\underline{T}$. So $(\underline{\overline{\mathbb{G}}}_m^t)^{\mathrm{val}}$ is a group object in the category of valuative log spaces over $\underline{T}$, or even in the smaller category

$$(\text{alg. val}/\underline{T})$$

of *algebraic valuative log spaces over* $\underline{T}$, i.e. valuative log spaces over $\underline{T}$ which are locally of the form $\underline{X}_{\mathrm{val}}$ for $\underline{X}$ a fine log scheme over $\underline{T}$, which is of finite type over T as a scheme. The trouble is that log spaces of the form $\underline{X}^{\mathrm{val}}$ are in general pretty big : the underlying space is seldom a scheme; locally $\underline{X}^{\mathrm{val}}$ is obtained as an inverse limit of fine log schemes under transition maps which are *log blow-ups* : a map $\underline{X} \to \underline{Y}$ of fine log schemes is called a log blow-up if locally on Y there is a chart $P \to \mathcal{O}_Y$ and a non empty ideal I of P ("ideal" means "$PI \subset I$") such that $X \to Y$ is the pull-back by $Y \to \operatorname{Spec} \mathbf{Z}[P]$ the blow-up map $\operatorname{Proj}(\oplus < I >^n) \to \operatorname{Spec} \mathbf{Z}[P]$ where $< I >$ is the ideal of $\mathbf{Z}[P]$ generated by I. However, while the "val" completion is difficult to grasp, maps in the category (alg.val/$\underline{T}$) are somehow easy to describe : indeed, any map $u : \underline{X}^{\mathrm{val}} \to \underline{Y}^{\mathrm{val}}$ can be locally realized as a "fraction" $\underline{X} \overset{s}{\leftarrow} \underline{X}' \to \underline{Y}$, where s is a log-blow-up.

Let us now return to the Tate curve. Kato shows that $(\underline{E}^t)^{\mathrm{val}}$ is a group object in (alg.val/$\underline{T}$) having the following properties : for any integer $n \geq 1$,

$$(\underline{E}^t)^{\mathrm{val}} \times_{\underline{T}} \underline{T}_n = (\underline{E}^t_n)^{\mathrm{val}} ,$$

where $\underline{E}^t_n = \underline{E}^t \times_{\underline{T}} \underline{T}_n$, and $(\underline{E}^t_n)^{\mathrm{val}}$ is the quotient

$$(\underline{E}^t_n)^{\mathrm{val}} = (\overline{\underline{\mathbf{G}}}^t_m)^{\mathrm{val}}_{\underline{T}_n}/\underline{t}^{\mathbf{Z}}$$

of the algebraic valuative space associated to $\overline{\underline{\mathbf{G}}}^t_m$ restricted to $\underline{T}_n$ by the discrete group $\mathbf{Z}$ acting on its $\underline{Z}$-valued points by $(i,a) \mapsto \underline{t}^i a$ (with $a \in \Gamma(Z, M_Z)$, $\underline{t}$ the image of $\underline{t} \in \Gamma(T, M_T)$ in $\Gamma(Z, M_Z)$ as above); on the generic point, $(\underline{E}^t)^{\mathrm{val}}$ induces the usual Tate curve E^t_K (with trivial log structure); the group structure of $(\underline{E}^t)^{\mathrm{val}}$ is compatible with the action of $(E^t)^0$ on E^t mentioned above.

Note that it is just for simplicity that we have taken for t a uniformizing parameter of T, the whole theory can be carried along assuming only that t is a non-zero element of the maximal ideal, or even, as far as the "formal" part is concerned, over an arbitrary base scheme T with a locally nilpotent global section t of $\mathcal{O}_T$ (the log structure on T being of course that associated to $\mathbf{N} \to \Gamma(T, \mathcal{O}_T)$, $1 \mapsto t$).

3.2. Log finite flat group schemes and log Barsotti-Tate groups.

3.2.1. Examples.

Let $\underline{S}$ be one of the log schemes $\underline{T}_N$ of 3.1, denote by $\underline{E}^t = (\underline{E}^t_{\underline{S}})^{\mathrm{val}}$ the Kato-Tate curve on $\underline{S}$ (a group object in (alg. val/$\underline{S}$)) associated to a non-zero element t of $\mathfrak{m}_T$. It is interesting to consider points of finite order of $\underline{E}^t$. If G is an abelian group and n an integer ≥ 1, denote by ${}_nG$ the kernel of the multiplication by n on G. As a group-valued functor on (alg. val/$\underline{S}$),

$$(3.2.1.1) \qquad {}_n\underline{E}^t$$

associates to $\underline{Z}$ the quotient

$$(\bigcup_{i \in \mathbf{Z}} \{a \in \Gamma(Z, M_Z) \mid a^n = \underline{t}^i\})/\underline{t}^{\mathbf{Z}} .$$

As an algebraic valuative log space over $\underline{S}$, it is associated to a fine log scheme over $\underline{S}$, still denoted ${}_n\underline{E}^t$, whose underlying scheme ${}_nE^t$ is the disjoint union

$${}_nE^t = \coprod S[u]/(u^n - t^i)$$

indexed by $i \in \mathbf{Z}/n\mathbf{Z}$, hence is in particular finite and flat over S. The group law

$${}_n\underline{E}^t \times_{\underline{S}} {}_n\underline{E}^t \to {}_n\underline{E}^t$$

is a priori defined only in the category (alg. val/$\underline{S}$). It turns out, however, that it is already defined in the simpler category

$$(3.2.1.2) \qquad (\text{alg.fs}/\underline{S})$$

of algebraic fine and *saturated* log schemes over $\underline{S}$ (an abstract monoid P is said to be saturated if it is integral and if $a \in P^{gp}$, then $a^n \in P$ for some integer $n \geq 1$ implies $a \in P$ (this is an analogue of normality); a log structure is said to be saturated if the stalks of the monoid are saturated; finally, "algebraic" means that the underlying scheme is locally of finite type over S); the point is that if $\underline{Z}$ is in (alg.fs/S), then for elements a, b of $\Gamma(Z, M_Z)$ such that $a^n = \underline{t}^i$, $b^n = \underline{t}^j$ for some i, j in $\mathbf{Z}$ and if $i + j = rn + s$ with $0 \leq s < n$, then one can replace ab by $ab/\underline{t}^r$, which belongs to $\Gamma(Z, M_Z)$ by the saturation property. Consider now the map from ${}_n\underline{E}^t$ to the constant group $\mathbf{Z}/n\mathbf{Z}$ on $\underline{S}$ associating to a $\underline{Z}$-valued point a of ${}_n\underline{E}^t$ the class in $\mathbf{Z}/n\mathbf{Z}$ of the integer i such that $a^n = \underline{t}^i$. This defines an exact sequence of group objects in (alg.fs/$\underline{S}$)

$$(3.2.1.3) \qquad 0 \to (\mathbf{Z}/n\mathbf{Z})(1) \to {}_n\underline{E}^t \to \mathbf{Z}/n\mathbf{Z} \to 0\,,$$

where $(\mathbf{Z}/n\mathbf{Z})(1)$ is the usual group scheme μ_n on S, endowed with the inverse image log structure. Taking the direct limit of these exact sequences on $\underline{S} = \underline{T}_N$ for $N \geq 1$ yields an exact sequence of group objects in the analogously defined category (alg.fs/$\underline{T}$)

$$(3.2.1.4) \qquad 0 \to (\mathbf{Z}/n\mathbf{Z})(1) \to {}_n\underline{E}^t \to \mathbf{Z}/n\mathbf{Z} \to 0\,,$$

where again $\mathbf{Z}/n\mathbf{Z}$ (resp. $(\mathbf{Z}/n\mathbf{Z})(1)$ denotes the usual group scheme $\mathbf{Z}/n\mathbf{Z}$ (resp. μ_n) over T, endowed with the inverse image log structure. One can check that over the generic point (3.2.1.4) induces the well known exact sequence

$$(3.2.1.5) \qquad 0 \to (\mathbf{Z}/n\mathbf{Z})(1) \to {}_nE^t \to \mathbf{Z}/n\mathbf{Z} \to 0\,,$$

defined in [13, VII 1.15]. But (3.2.1.4) is quite a new object, since in general there is no exact sequence of finite flat group schemes over T extending (3.2.1.5) (the only case where such an extension exists is when t is of valuation n, in which case (3.2.1.4), on the underlying schemes, coincides with the exact sequence (VII 1.13.2) of (loc. cit.) (the special fiber of E^t is then a Néron n-gon)). Finally, if p is a fixed prime number, the exact sequences (3.2.1.4) for $n = p^r$, $r \geq 1$, form an inductive system, whose limit is an exact sequence of ind-objects of (alg.fs/T)

$$(3.2.1.6) \qquad 0 \to (\mathbf{Q}_p/\mathbf{Z}_p)(1) \to {}_{(p^\infty)}\underline{E}^t \to \mathbf{Q}_p/\mathbf{Z}_p \to 0\,,$$

where $\mathbf{Q}_p/\mathbf{Z}_p$ and $(Q_p/\mathbf{Z}_p)(1)$ are the usual Barsotti-Tate groups over T.

3.2.2. **Structure of log finite flat group schemes and log Barsotti-Tate groups.** The objects ${}_n\underline{E}^t$ in (3.2.1.1) and (3.2.1.3) are examples of log finite flat group schemes, and the object ${}_{p^\infty}\underline{E}^t$ in (3.2.1.6) is an example of a log Barsotti-Tate group. Kato has taken up a systematic study of such objects and already obtained important structure theorems. Some are given in [29], others will appear in forthcoming papers. See also Messing's report in this volume [38]. For lack of space, I will limit myself to a very brief discussion.

Fix a fine and saturated log scheme $\underline{T}$, whose underlying scheme T is locally noetherian. Consider the category of fine and saturated log schemes $\underline{X}$ over $\underline{T}$, whose underlying scheme is locally of finite type, which category we will simply denote here

$$(\mathrm{fs}/\underline{T}) .$$

In contrast with the classical case, there are at least two natural notions of log finite flat group schemes over $\underline{T}$. A log finite flat group scheme on $\underline{T}$ *in the wide sense* is a commutative group object $\underline{G}$ in $(\mathrm{fs}/\underline{T})$, which is log flat and small over $\underline{T}$ ([1]) and whose underlying scheme is finite over T; when the Cartier dual of $\underline{G}$ (i.e. the sheaf $\underline{\mathrm{Hom}}(\underline{G}, \mathbb{G}_m)$ on the flat site $\underline{T}_{fl}$ of $(\mathrm{fs}/\underline{T})$ is representable by an object with the same properties, $\underline{G}$ is said to be a log finite flat group scheme on $\underline{T}$ *in the restrictive sense* (the log flat site is generated by covering families $(f_i : \underline{X}_i \to \underline{X})$ such that f_i is log flat, exact, and small, and $\cup f_i(X_i) = X$). In what follows, we shall consider only the restrictive notion, and simply call the corresponding $\underline{G}$*'s log finite flat group schemes over* $\underline{T}$. If p is a prime number, a *log p-divisible group* (or *Barsotti-Tate group*, or *BT group*) over $\underline{T}$ is a sheaf G of abelian groups on $\underline{T}_{fl}$, which is of p-torsion and is p-divisible, and such that each ${}_{(p^n)}G$ is representable by a log finite flat group scheme over $\underline{T}$. Kato has given two types of structure theorems : (a) for log finite flat group schemes over local (artinian or complete) bases $\underline{T}$ such that $M_T/\mathcal{O}_T^* = \mathbf{N}$, (b) for log BT's over local or global bases (Dieudonné theory).

(a) **Structure of log finite flat group schemes** ([2]).

Assume that T is the spectrum of a complete noetherian local ring and that $M_T/\mathcal{O}_T^* = \mathbf{N}$. Let $\underline{G}$ be a log finite flat group scheme (in the restrictive sense) over $\underline{T}$. Kato shows that $\underline{G}$ has a largest log étale quotient $\underline{G}^{\text{ét}}$, which is in fact étale in the classical sense and has the inverse image log structure ([3]) . Moreover, if $\underline{G}^0$ is the kernel of $\underline{G} \to \underline{G}^{\text{ét}}$, G^0 is the connected component of the origin and the log structure is the inverse image of that of $\underline{T}$. Thus we have an exact sequence of log finite flat group schemes over $\underline{T}$

$$0 \to \underline{G}^0 \to \underline{G} \to \underline{G}^{\text{ét}} \to 0 , \tag{3.2.2.1}$$

where $\underline{G}^{\text{ét}}$ and $\underline{G}^0$ are usual finite flat group schemes over T, respectively étale and connected, equipped with the inverse image log structures. Kato shows that, if one fixes

([1]) A map $f : \underline{X} \to \underline{Y}$ of fine log schemes is (log) *flat* if locally it admits a chart c as in 1.2 such that $\operatorname{Spec} \mathcal{O}_X[P^{gp}] \to \operatorname{Spec} \mathcal{O}_X[Q^{gp}]$ and $X \to Y \times_{\operatorname{Spec} \mathbf{Z}[Q]} \operatorname{Spec} \mathbf{Z}[P]$ are flat. It is *small* if for any $x \in X$, the cokernel of $(f^*M_Y)^{gp}_{\overline{x}} \to M^{gp}_{X,\overline{x}}$ is a torsion group.

([2]) Most of the results of this section are proven in [31] in the slightly different context of valuative log spaces. Complete proofs are to appear in other papers of Kato.

([3]) This last assertion is not proven in [31]; see [31, 5.2.10] for a converse.

an element $\underline{t}$ of $\Gamma(T, M_T)$ whose image in $M_T/\mathcal{O}_T^*$ is a generator, then (3.2.2.1) amounts to the data of a classical extension of finite flat group schemes over T

$$0 \to G^0 \to E \to G^{\text{ét}} \to 0 \tag{3.2.2.2}$$

(so that $G^{\text{ét}}$ (resp. G^0) is the étale quotient $E^{\text{ét}}$ (resp. the connected component of the origin E^0) of E and of a "monodromy map"

$$N : G^{\text{ét}}(1) \to G^0 \ , \tag{3.2.2.3}$$

where $G^{\text{ét}}(1)$ is the (connected) group which is the Cartier dual of the Pontrjagin dual of $G^{\text{ét}}$. In fact, (3.2.2.1) can be recovered as the sum (in the group of extensions of $\underline{G}^{\text{ét}}$ by $\underline{G}^0$ over $\underline{T}$) of the inverse image on $\underline{T}$ of (3.2.2.2) and the push-out by N of an exact sequence of the form (3.2.1.3), twisted by $\underline{G}^{\text{ét}}$. More precisely, Kato construct an equivalence of categories between the category of log finite flat group schemes over $\underline{T}$ and the category of pairs (E, N) consisting of a usual finite flat group scheme E over T and a map $N : E^{\text{ét}}(1) \to E^0$.

Suppose in particular that T is the spectrum of a perfect field of characteristic $p > 0$, so that $\underline{T}$ is a standard log point. Then the extension (3.2.2.2) splits. Combining with classical Dieudonné theory, one then obtains an anti-equivalence between the category of log finite flat group schemes over $\underline{T}$ and the category of 4-uples (M, F, V, N), where M is a $W(k)$-module of finite length, F (resp. V) a σ-linear (resp. σ^{-1}-linear) endomorphism of M (where σ is the Frobenius automorphism of $W(k)$) and N is a linear endomorphism of M, F, V, N being subject to the relations $FV = VF = p$ and $FNV = N$. Note that automatically $N^2 = 0$. For log $BT's$ over $\underline{T}$, one obtains an analogous (anti)-equivalence, with "finite length" replaced by "free of finite type". Such objects (M, F, V, N), with M not necessarily of finite type, had already been encountered in a quite different context in [27]; it was an elementary observation in (loc. cit.) that for M of finite type, the topological nilpotence of V forces N to be zero. In the above situation, N thus has to factor through the semi-simple part under V of M, which of course corresponds to the Dieudonné module of the multiplicative part of G^0. As an (easy) application of the above dictionary, one can show that if k is algebraically closed, then any log BT over $\underline{T}$ is a sum of usual $BT's$ over T and copies of the log BT of the special fiber of the Kato-Tate curve $\underline{E}^{p^a}$ over $W(k)$ with invariant p^a, $a \geq 1$.

(b) Structure of log BT groups.

Kato has generalized the Dieudonné theory of Grothendieck ([21], [22]) and Berthelot-Breen-Messing ([4], [5], [6]). If $\underline{T}$ is a fine and saturated log scheme which is log smooth and of finite type over the spectrum of a perfect field k of characteristic p, Kato constructs a (contravariant) Dieudonné functor D from the category of log BT groups over $\underline{T}$ to the category of Dieudonné crystals on $\underline{T}/W(k)$; a Dieudonné crystal is a crystal M on $\underline{T}/W(k)$, which is locally free of finite type, and which is equipped with maps $F : \mathrm{Frob}^*M \to M$, $V : M \to \mathrm{Frob}^*M$, where Frob is the absolute Frobenius, satisfying $FV = p$, $VF = p$. Kato shows that this functor is an equivalence of categories. Forgetting the log structures, or rather specializing to the case of trivial log structures, gives, with a little more work, the missing essential surjectivity statement in [6]. Kato's method uses the extra assumption $p \neq 2$, but Messing has recently been able to remove it, see his report in this volume.

3.3. Log abelian schemes.

Let $\underline{T}$ be a log scheme satisfying the hypotheses of 3.2.2. One can define a *log abelian scheme* over $\underline{T}$ as a group object $\underline{X}$ in the category of algebraic valuative log schemes over $\underline{T}$ (cf. 3.1), which is associated to a fine, saturated, smooth log scheme over $\underline{T}$, whose underlying scheme is proper over T. A typical example is the Kato-Tate curve $\underline{E}^t_{\underline{T}}$. More generally, with $\underline{T}$ as in 3.1, if X_K is an abelian scheme over K having semi-stable reduction, Kato can construct a log abelian scheme $\underline{X}$ over $\underline{T}$ extending X_K; for $n \geq 1$, ${}_n\underline{X}$ is then a log finite flat group scheme extending ${}_nX_K$, and ${}_{p^\infty}\underline{X}$ a log BT group extending ${}_{p^\infty}X_K$. Combining with the Dieudonné theory of 3.2.2 (b), Kato is able to generalize Bauer's results [1] to the case of abelian schemes over the function field of a proper smooth curve over a finite field having everywhere semi-stable reduction. All these results are to appear shortly, we hope.

This, however, should not conceal the fact that the theory of log abelian schemes is still in infancy : the most basic questions are open, and it is not even clear that the above definition is the right one. For example :

(a) Is a log abelian scheme automatically commutative?

(b) Is a pointed map of log abelian schemes automatically a map of groups, in other words is there an analogue of the well known rigidity theorem?

(c) What is the dual of a log abelian scheme?

Question (c) raises in fact more fundamental questions, such as defining a line bundle (or more generally a vector bundle, or a coherent sheaf) on a log scheme $\underline{S}$. A line bundle on $\underline{S}$ should have a class in $H^1(S_{\text{ét}}, M_S^{gp})$, since this group seems to be the right substitute for the Picard group. Kato has calculated the corresponding "Picard functor" for the Kato-Tate curve [31, 2.2.4], and one can expect this to be just the beginning of a nice general theory, encompassing in particular the theory of "cubical sheaves" of Breen [7] and Moret-Bailly [39].

Since Chai-Faltings's compactifications of A_g [8] involve toroidal embeddings, one might also hope that they can be obtained as solutions of suitable moduli problems on log schemes. One can of course have similar expectations for other moduli problems, such as curves [12] or K3 surfaces (see e.g. [20]). But although a nice deformation theory for log spaces seems to exist (Bauer-Illusie, work in progress), one is still a far cry from giving an adequate answer to these questions.

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