# Prismatic $F$-gauges and Frobenius-liftable schemes

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## Abstract

We give a description of prismatic $F$-gauges on Frobenius liftable schemes in terms of big Fontaine-Laffaille modules. The main conceptual insight is that the Nygaard filtered prismatization of such a scheme is pulled back from the Hodge-filtered de Rham stack of its lift. Along the way we study a notion of $p^n$-de Rham stacks which provide a lift of prismatic cohomology to $\mathbf{Z}/p^{n+1}$-schemes, with the case $n = 1$ being relevant to our study.

## Prismatic $F$-gauges as $p$-adic motives

Fix a prime $p$ and let $k$ be a perfect field of characteristic $p$. Work of Bhatt–Lurie and Drinfeld [Bha22, BL22a, BL22b, Dri24] associates to any reasonable $k$-scheme $Y$, a $p$-complete and $W(k)$-linear symmetric monoidal stable $\infty$-category $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$, of **prismatic $F$-gauges on** $Y$, which one may think of as $p$-adic étale (mixed) motives on $Y$.

This category has the following features

1. There is a unit object $\mathbf{1}_Y$ and for each $n \in \mathbf{Z}$ an invertible endofunctor $(-)\{n\}$ so that
$$\mathrm{Ext}^i_{F\text{-Gauge}_{\mathbbl{\Delta}}(Y)}(\mathbf{1}_Y, \mathbf{1}_Y\{n\}) = H^i_{\mathrm{Syn}}(Y, \mathbf{Z}_p(n))$$
where the right hand side is the weight $n$ syntomic cohomology of [BMS19] (already known to Milne in our setting [Mil76]) and can be regarded as $p$-adic étale motivic cohomology [BM23].
2. If $X \to Y$ is a morphism of schemes then there is a canonical complex $\mathcal{H}_{\mathrm{Syn}}(X) \in F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ so that $\mathrm{Ext}^i_{F\text{-Gauge}_{\mathbbl{\Delta}}(Y)}(\mathbf{1}_Y, \mathcal{H}_{\mathrm{Syn}}(X)\{n\})$ is closely related to the syntomic cohomology of $X$ via a Leray spectral sequence.
3. There are realization functors from $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ to other coefficient systems for various $p$-adic cohomology theories (e.g. crystalline, prismatic, Hodge-Tate and de Rham along with their filtrations). Moreover, it contains étale local systems as a full subcategory.

Thus $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ provides a robust notion of coefficients for syntomic cohomology.

## Main motivation: Relation to $p$-adic (étale) $K$-theory

A deep motivic property of the complexes $\mathbf{Z}_p(n)$ of [BMS19],[CMM21] is that they provide a filtration on $p$-adic (étale) K-theory. More concretely there is a spectral sequence
$$H^s_{\mathrm{Syn}}(Y, \mathbf{Z}_p(d)) \Rightarrow K_{2d-s}(Y)^{\wedge}_p.$$
Thus the category $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ contains deep $K$-theoretic information not just about $Y$, but also for schemes over $Y$. Thus it would be desirable to describe $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ in certain specific cases.

## Cohomological stacks

The association $Y \mapsto F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ factors through an intermediate association of a $p$-adic formal stack $Y \mapsto Y^{\mathrm{Syn}}$, the so-called **syntomification of** $Y$, so that:

1. By definition, $F\text{-Gauge}_{\mathbbl{\Delta}}(Y) := \mathcal{D}_{\mathrm{qc}}(Y^{\mathrm{Syn}})$,
2. There is a canonical line bundle $\mathcal{O}\{1\}$ so that $H^i(Y^{\mathrm{Syn}}, \mathcal{O}\{n\}) = H^i_{\mathrm{Syn}}(Y, \mathbf{Z}_p(n))$, where $\mathcal{O}\{n\} = \mathcal{O}\{1\}^{\otimes n}$; in other words the endofunctor $(-)\{n\}$ is just tensoring with the appropriate line bundle.
3. For any morphism $f\colon X \to Y$, the $F$-gauge $\mathcal{H}_{\mathrm{Syn}}(X) := Rf^{\mathrm{Syn}}_* \mathcal{O}_{X^{\mathrm{Syn}}}$ for the associated morphism of $p$-adic formal stacks $f^{\mathrm{Syn}}\colon X^{\mathrm{Syn}} \to Y^{\mathrm{Syn}}$.

The geometry of $Y^{\mathrm{Syn}}$ therefore controls the structure of $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$.
In fact, the association $Y \mapsto Y^{\mathrm{Syn}}$ is mediated by two further associations of stacks:

1. the association $Y \mapsto Y^{\mathbbl{\Delta}}$, the **prismatization** of $Y$ whose quasi-coherent sheaves are prismatic crystals,
2. and the association $Y \mapsto Y^{\mathrm{Nyg}}$ the **Nygaard filtered prismatization**, whose structure morphism to $\mathrm{Spf}(W(k))$ factors over $\mathbf{A}^1/\mathbf{G}_m$, and so the quasi-coherent sheaves carry the Nygaard filtration on prismatic crystals.

The stack $Y^{\mathbbl{\Delta}}$ sits inside $Y^{\mathrm{Nyg}}$ via two maps $j_{\mathrm{dR}}, j_{\mathrm{HT}}\colon Y^{\mathbbl{\Delta}} \hookrightarrow Y^{\mathrm{Nyg}}$, the so-called **de Rham and Hodge-Tate loci**, which in some sense witness the fact that prismatic cohomology is a Frobenius descent of crystalline cohomology.
The stack $Y^{\mathrm{Syn}}$ is then defined as
$$Y^{\mathrm{Syn}} := \mathrm{coeq}(Y^{\mathbbl{\Delta}} \underset{j_{\mathrm{HT}}}{\overset{j_{\mathrm{dR}}}{\rightrightarrows}} Y^{\mathrm{Nyg}})$$

## Goal

Our goal is to use the geometry of the aforementioned cohomological stacks $Y^{\mathbbl{\Delta}}, Y^{\mathrm{Nyg}}$ and $Y^{\mathrm{Syn}}$ to describe $F\text{-Gauge}_{\mathbbl{\Delta}}(Y)$ in simple situations. More precisely, we will completely describe all three stacks when $Y$ is smooth and admits a lift to a $p$-adic formal scheme $\mathfrak{Y}$ over $W(k)$ *along with* a $W(k)$-linear endomorphism $\varphi_{\mathfrak{Y}}$ which lifts the Frobenius on $Y$. We can also say something for mildly singular schemes with a Frobenius lift to $W(k)$, which we will omit for the sake of brevity.

## The Frobenius-liftable case

Let $\mathfrak{Y}/W(k)$ be a smooth $p$-adic formal scheme. In the sequel we will say that $\mathfrak{Y}/W(k)$ is a $\delta$-scheme if it carries a lift of the Frobenius (this is equivalent to carrying a $\delta$-structure in the smooth case, which is our setting).

Associated to $\mathfrak{Y}/W(k)$ there are three 'de Rham stacks':

- $\mathfrak{Y}^{\mathrm{dR},+}$, which lives over $\mathbf{A}^1/\mathbf{G}_m$ and parametrises Hodge filtered de Rham cohomology,
- $\mathfrak{Y}^{\mathrm{dR}}$ which parametrises de Rham cohomology relative to $W(k)$; it is the specialization of $\mathfrak{Y}^{\mathrm{dR},+}$ to the locus $\mathbf{G}_m/\mathbf{G}_m$.
- $\mathfrak{Y}^{p\text{-dR}}$, **the $p$-de Rham stack of** $\mathfrak{Y}$, parametrising a variant of de Rham cohomology where one scales the differential by $p$; it is the specialization of $\mathfrak{Y}^{\mathrm{dR},+}$ at the point $i_p\colon \mathrm{Spf}(W(k)) \to \mathbf{A}^1/\mathbf{G}_m$ corresponding to the coordinate $t = p$

If $\mathfrak{Y}/W(k)$ is a $\delta$-scheme, then there is a 'divided Frobenius' morphism
$$\frac{F}{p}\colon \mathfrak{Y}^{\mathrm{dR}} \to \mathfrak{Y}^{p\text{-dR}},$$
which is an isomorphism of stacks.

## Strategy

Let $\mathfrak{Y}/W(k)$ be a smooth $\delta$-scheme. Study the syntomification of the special fiber $Y := \mathfrak{Y} \otimes_{W(k)} k$ in terms of the structures on the aforementioned de Rham stacks.

## The Nygaard filtered prismatization of a Frobenius liftable scheme is pulled back.

In [Sah26b], our first main observation is the following:

### Theorem: pulling back the Nygaard filtered prismatization

**Theorem** (S., Petrov-Vologodsky [Sah26b]) Let $\mathfrak{Y}/W(k)$ be a smooth $\delta$-scheme and $Y := \mathfrak{Y} \otimes_{W(k)} k$. Then there is a pullback square
$$\begin{tikzcd} Y^{\mathrm{Nyg}} \arrow[r] \arrow[d] & \mathfrak{Y}^{\mathrm{dR},+} \arrow[d] \\ k^{\mathrm{Nyg}} \arrow[r] & \mathbf{A}^1/\mathbf{G}_m \end{tikzcd}$$
This pullback is functorial in the $\delta$-scheme $\mathfrak{Y}/W(k)$.

**Proof sketch:** The first proof of this result, due to the author, hinges on an observation of [BMS19] (going back to Illusie) which says that for a $\delta$-scheme $\mathfrak{Y}/W(k)$ the Nygaard filtration on the crystalline cohomology of the special fiber $Y$ is just the tensor product of the Hodge filtration and the $p$-adic filtration on the de Rham cohomology of $\mathfrak{Y}/W(k)$. To lift this statement, which is at the level of cohomology, to one for stacks, we use a notion of *affineness in the derived sense* due originally to Toën [Toë06] and recast into modern form in [MM25].

More precisely, we show in [Sah25] the following theorem (stated a bit heuristically to avoid technicalities)

### Theorem: de Rham affineness

**Theorem**(S. [Sah25]): Let $Y$ be any affine scheme over $k$. Then $Y^{\mathrm{Nyg}}$ viewed as a stack over $k^{\mathrm{Nyg}}$ is isomorphic to the relative spectrum of the Nygaard filtered prismatic cohomology of $Y$. This isomorphism is functorial in $Y$.

A similar result is shown for the stack $\mathfrak{Y}^{\mathrm{dR},+}$ as a stack over $\mathbf{A}^1/\mathbf{G}_m$ in [Sah26b]. The functorial affineness which we dub **de Rham affineness**, then formally determines the displayed pullback square.

Making this theory work involved making sense of **non-connective animated rings over formal stacks**. This work is carried out in [Sah26a].

After some discussions with the author, Petrov-Vologodsky give a different proof of the pullback diagram which involves a deeper study of the notion of ring stacks with $\delta$-structures which we skip to avoid technicalities.

## Gauges as filtered $\mathcal{D}$-modules

Let $\mathfrak{Y}/W(k)$ be a smooth $\delta$-scheme. $\mathcal{D}^{\mathrm{crys}}_{\mathfrak{Y}}$ be the sheaf of crystalline differential operators i.e. the universal enveloping algebroid of the tangent bundle of $\mathfrak{Y}$ . Equip it with the **Nygaard filtration** $\mathrm{Fil}_{\mathrm{Nyg}}(\mathcal{D}^{\mathrm{crys}}_{\mathfrak{Y}})$, defined as the tensor product of the order filtration and the $p$-adic filtration.

An immediate pay off of the previous theorem is the following

### Theorem: Gauges as filtered $\mathcal{D}_{\mathfrak{Y}}$-modules

**Theorem**(S. [Sah26b]) Let $\mathfrak{Y}/W(k)$ be a smooth $\delta$-scheme with special fiber $Y$. There is an equivalence
$$\mathcal{D}_{\mathrm{qc}}(Y^{\mathrm{Nyg}}) \simeq \mathrm{FilMod}^{\mathrm{lnilp}}_{p\text{-comp}}(\mathrm{Fil}_{\mathrm{Nyg}}(\mathcal{D}^{\mathrm{crys}}_{\mathfrak{Y}})),$$
where the right-hand side consists of derived $p$-complete filtered complexes with locally nilpotent $p$-curvature modulo $p$.

## Big Fontaine–Laffaille modules

The Rees construction applied to $\mathrm{Fil}_{\mathrm{Nyg}}(\mathcal{D}^{\mathrm{crys}}_{\mathfrak{Y}})$, along with the pushout description of $Y^{\mathrm{Syn}}$, and with the interactions of the de Rham stacks described previously, one gets the following concrete description of prismatic $F$-gauges on a Frobenius liftable scheme:

### Theorm: Fontaine–Laffaille description

**Theorem**(S. [Sah26b]) Let $\mathfrak{Y}/W(k)$ be a smooth $\delta$-scheme and $Y$ the special fiber. Objects of $\mathcal{D}_{\mathrm{qc}}(Y^{\mathrm{Syn}})$ correspond to graded, derived $p$-complete modules $M^{\bullet}$ over
$$\mathrm{Rees}(\mathrm{Fil}_{\mathrm{Nyg}}(\mathcal{D}^{\mathrm{crys}}_{\mathfrak{Y}}))$$
with locally nilpotent $p$-curvature modulo $p$, and equipped with an isomorphism
$$F^*M^{\bullet}\big|_{t=p} \simeq M^{\bullet}\big|_{t=1}.$$

This is a relative 'big Fontaine–Laffaille' theory, in the sense of [TVX25]: there is no restriction on Hodge–Tate weights.

## Recovering a result of Ogus on the comparison between prismatic crystals and $p$-connections

In recent work [Ogu24], Ogus has given a description of prismatic crystals in terms of $p$-connections. We are able to recover a stacky version of this result.

### Theorem: recovering Ogus

**Theorem**(Ogus [Ogu24], [Sah26b]) Let $\mathfrak{Y}/W(k)$ be a smooth $\delta$-scheme with special fiber $Y$. Then there is an equivalence of $p$-adic formal stacks over $\mathrm{Spf}(W(k))$
$$\mathfrak{Y}^{p\text{-dR}} \simeq Y^{\mathbbl{\Delta}}.$$
In particular there is an equivalence of stable $\infty$-categories
$$\mathrm{Crys}_{\mathbbl{\Delta}}(Y) \simeq \mathcal{D}_{\mathrm{qc}}(\mathfrak{Y}^{p\text{-dR}})$$
where the left hand side is prismatic crystals and the right hand side is just $p$-connections.

## A mod $p^n$ lift of prismatic cohomology and a mystery

Let $\mathfrak{Y}/W(k)$ be any smooth $p$-adic formal scheme. We have defined the $p$-de Rham stack $\mathfrak{Y}^{p\text{-dR}}$.

### $p^2$-crystallinity of the $p$-de Rham stack

For any smooth $p$-adic formal scheme $\mathfrak{Y}/W(k) \mapsto \mathfrak{Y}^{p\text{-dR}}$ factors through the mod $p^2$-fibre i.e. $\mathfrak{Y}/W(k) \mapsto \mathfrak{Y} \otimes_{W(k)} W(k)/p^2$.

As a consequence there is a 'new' cohomology theory on $W(k)/p^2$-schemes, valued in (derived) $p$-complete $W(k)$-modules. Explicitly, this sends any smooth $W(k)/p^2$-scheme $Y$, to the cohomology $Y \mapsto H^*(Y^{p\text{-dR}}, \mathcal{O})$. This assignment depends on the scheme $Y$ and **not just the special fibre**.

In fact for any $n > 0$ there is a $p^n$-de Rham stack

### Theorem/ Expectation: mod $p^n$-twists of prismatic cohomology

For every $n > 0$ there is a 'new' cohomology theory on $W(k)/p^{n+1}$-schemes with values in (derived) $p$-complete $W(k)$-modules. Explicitly for every smooth $Y \in \mathrm{Sch}_{W(k)/p^{n+1}}$ there is a stack
$$Y^{p^n\text{-dR}} \longrightarrow \mathrm{Spf}(W(k))$$
so that the cohomology theory is $Y \mapsto H^*(Y^{p^n\text{-dR}}, \mathcal{O})$. This cohomology theory is uniquely characterized as a functorial lift of the $p^n$-de Rham complex:
$$R\Gamma(Y^{p^n\text{-dR}}, \mathcal{O}) \otimes^{\mathbf{L}} W(k)/p^{n+1} \simeq R\Gamma\big(Y, \mathcal{O}_Y \xrightarrow{p^n d} \Omega^1_Y \xrightarrow{p^n d} \Omega^2_Y \to \cdots\big).$$
When $Y$ is a mod $p^{n+1}$ Frobenius lift of its special fibre $Y_0$, then this cohomology theory is **just the prismatic cohomology of** $Y_0$. **In general this theory does not depend only on the special fibre.**

The author would like to know if there is a good reason for these deformations to exist.

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